Spectral kissing and its dynamical consequences in the squeezed Kerr-nonlinear oscillator

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Transmon qubits are the predominant element in circuit-based quantum information processing due to their controllability and ease of engineering implementation. But more than qubits, transmons are multilevel nonlinear oscillators that can be employed in the discovery of new fundamental physics. Here, they are explored as simulators of excited state quantum phase transitions (ESQPTs), which are generalizations of quantum phase transitions to excited states. We show that the coalescence of pairs of adjacent energy levels (spectral kissing) recently observed with a squeezed Kerr oscillator [arXiv:2209.03934] is an ESQPT precursor. The classical limit of this system explains the origin of the quantum critical point and its consequences for the quantum dynamics, which includes both the fast scrambling of quantum information, characterized by the exponential growth of out-of-time-ordered correlators, and the slow evolution of the survival probability at initial times, caused by the localization of the energy eigenstates at the vicinity of the ESQPT. These signatures of ESQPT in the spectrum and in the quantum dynamics are simultaneously within reach for current superconducting circuits experiments.

Recent developments in superconducting circuits have opened the pathway to explore long standing predictions of quantum physics. They have been used to study dynamical bifurcation [11][2], to squeeze quantum fluctuations [3], to prepare exotic quantum states, and to process and stabilize quantum information [1][5]. Here, we propose to use this platform as a quantum simulator of excited state quantum phase transitions (ESQPTs), a phenomenon that occurs in various systems in nuclear, atomic, molecular, and condensed matter physics.

A quantum phase transition (QPT) corresponds to an abrupt change in the ground state of a physical system when a control parameter reaches a critical point. It occurs in the thermodynamic limit, but scaling analyses of finite systems can signal its presence. ESQPT is a generalization of this phenomenon to excited states [6][9], which can take place independently of the presence of QPTs [10][12] and can be triggered by anharmonicities [13][14]. In an ESQPT, the separation of the states in two phases [16] occurs at a point that depends on both the value of the energy and of the control parameter. There is a vast literature on the subject, which is reviewed in [9]. ESQPTs are associated with enhanced decoherence [17][18], localized eigenstates [19][21], very slow [19][21] or accelerated [22][24] quantum quench dynamics, particular dynamical features at long times [25][27], isomerization reactions [28], and the creation of Schrödinger cat states [12].

The main signatures of ESQPTs are the closing of energy gaps between excited states and a singularity in the density of states (DOS) that moves to higher excitation energies as the control parameter increases. The energy where the divergence of the DOS takes place is the ESQPT critical energy. These and related features have been theoretically identified in various quantum systems with few degrees of freedom [6][11], and detecting the ESQPT with spinor Bose-Einstein condensates has been proposed [12].

Even though spectroscopic signatures of the ESQPT have been experimentally observed [13][47] and its presence suggested from the bifurcation phenomenon detected in [48][50], presently none of these systems provides the means to observe the dynamical consequences of an ESQPT in a controllable way. Superconducting circuits offer a unique platform to research ESQPTs, because they have an experimental realizable classical limit, and they provide both frequency- and time-resolved high quantum non-demolition measurements fidelity [51].

As we explain here, the exponential approach of pairs of adjacent levels (“kissing”) recently observed in the spectrum of a superconducting Kerr oscillator as a function of the amplitude of a squeezing drive [51] marks the presence of an ESQPT. The dynamical counterpart of this transition presents a seeming paradoxical behavior, which can, in principle, be observed in a system such as the one in Ref. [51]. The evolution of Glauber coherent
states shows that the closest states to the ESQPT exhibit the slowest decays of the survival probability (overlap of the initial and the evolved state), but they also present the fastest scrambling. How can these apparently opposite behaviors coexist for the same range of excited states?

The answer comes from the analysis of the classical limit of the system. It reveals an unstable stationary point at the origin of the phase space, \( (q = 0, p = 0) \), where the evolution is dominated by the squeezing part of the Hamiltonian. For initial states close to this point, the variance of the phase-space distribution spreads at a rate determined by the Lyapunov exponent, as captured by the exponentially fast growth of the fidelity out-of-time-ordered correlator (FOTOC), and at the same time, a portion of the population remains localized at the origin for some time, as measured by the very slow evolution of the survival probability. The accelerated spread in phase space is followed by partial revivals, and this yo-yo effect persists as long as dissipative is negligible.

Quantum system.– The system that we analyze was implemented in a superconducting circuit \[51\] based on SNAIL \[52\] transmons. The Hamiltonian is given by \[53\]

\[
\hat{H}_{qu} = \hat{a}(\hat{a}^\dagger - 1) - \xi (\hat{a}^4 + \hat{a}^2),
\]

where \( \hat{a} = \hat{a}^\dagger \hat{a} \), \( K \) is the Kerr nonlinearity, \( \xi = \epsilon_2/K \) is the control parameter, and \( \epsilon_2 \) is the squeezing amplitude. The system conserves parity, \( [\hat{H}_{qu}, (-1)^{\hat{a}^\dagger \hat{a}}] = 0 \).

We study the spectrum of \( \hat{H}_{qu} \) as a function of the control parameter \( \xi \) in Figs. 1(a)-(e). The plots display the excitation energies, \( E' = (E - E_0) \), where \( E \) are the eigenvalues of \( \hat{H}_{qu} \) and \( E_0 \) its ground state energy. The numerical data in Fig. 1(a) reproduce the experimental data in Fig. 3A of Ref. \[51\]. One sees that as the control parameter increases, the separation between pairs of adjacent eigenvalues belonging to different parity sectors vanishes (spectral kissing) at higher energies. This becomes better visible in Fig. 1(b), where larger values of \( \xi \) are used. For a given value of the control parameter, the kissing happens at the critical energy of the ESQPT, \( E'_{\text{ESQPT}} \), which is marked with a solid line in Fig. 1(b) and is obtained analytically [see Eq. (3) below].

In addition to kissing, the eigenvalue clusters at \( E'_{\text{ESQPT}} \) \[53\]. This results in the peak of the DOS displayed for different values of the control parameter in Figs. 1(c)-(e). The peak diverges for \( \xi \to \infty \), which is a main signature of the ESQPT.

The presence of the ESQPT gets reflected in the structure of the eigenstates, \( |\psi\rangle = \sum_n C_n |n\rangle \), written in the Fock basis, \( \hat{a}^\dagger |n\rangle = n |n\rangle \). The eigenstates at the vicinity of the ESQPT are highly localized in the Fock state \( |0\rangle \). This can be quantified with the participation ratio, \( P_R = 1/\sum_{n=0}^{N-1} |C_n|^4 \), where \( N \) is the size of the truncated Hilbert space. \( P_R \) is large for an extended state and small for a localized state. In Figs. 1(f)-(h), we show the participation ratio as a function of \( E' \). An abrupt dip in the value of \( P_R \) happens for \( E' \sim E'_{\text{ESQPT}} \) and the analysis of the components of the eigenstate at this energy confirms its localization at \( |0\rangle \).

The localization at the ESQPT critical point is also detected with the Husimi quasiprobability distribution \[54\] obtained by writing the eigenstates in the Glauber coherent states \[55\]. It gives the distribution of the quantum state in the phase space of canonical variables \( (q, p) \). As seen in Fig. 1(k), the eigenstate closest to the ESQPT energy is highly concentrated in the origin of the phase space \( q, p = 0 \). This contrasts with the eigenstates below the ESQPT [Figs. 1(i)-j)], which present two separated ellipses, and the eigenstates above it [Fig. 1(l)]. The localization in the phase space mirrors the localization in the Fock basis, since the coherent state with \( q, p = 0 \) coincides with the Fock state \( |0\rangle \).
Classical limit.– The Hamiltonian for the Kerr nonlinear oscillator develops two wells when the squeezing drive is turned on, as evident from the Husimi distributions in Fig. 1 (i)-(l). The depth of the wells and their energy levels grow as $\xi$ increases, bringing the system closer to the classical limit. Experimentally, the value of $\xi$ can be increased by reducing the impedance of the circuit, increasing the microwave power of the squeezing drive, or approaching the Kerr-free point.

The grounds for the onset of the ESQPT are found in the classical limit. The classical Hamiltonian

$$H_{cl} = \frac{1}{2} (q' + p')^2 - \xi (q'^2 - p'^2),$$

(2)

presents three critical points where $\xi > 0$. They are the two center points $\{q, p\} = \{\pm \sqrt{2}\xi, 0\}$ with the minimal energy of the system $E_{\text{min}} = H_{cl}(q, p) = -K\xi^2$, and the hyperbolic point $\{q, p\} = \{0, 0\}$, which is a saddle point with energy $E_{\text{hyp}} = 0$. In the plot of the energy contours in Fig. 2 (a), the hyperbolic point is O, the line that intersects at this point is the separatrix, and the two blue diamonds are the center points.

The properties of the quantum system find a parallel in the classical limit. The energy difference $E_{\text{hyp}} - E_{\text{min}}$ marks the separatrix in Fig. 2 (a) and determines the energy of the ESQPT,

$$E_{\text{ESQPT}} = K\xi^2,$$

(3)

which is indicated with a bright orange line in Fig. 1(b).

Below this energy, the pairs of stable periodic orbits with equal energy are analogous to the degenerate states of the quantum system, and above that line the degeneracy is lost. The stationary point at the origin of the phase space, $(q, p) = (0, 0)$, justifies the localization at $|0\rangle$ of the eigenstate with energy at the ESQPT.

The existence of a non-degenerate hyperbolic point implies the logarithmic discontinuity of the level density[55], and explains the peak at $E'_{\text{ESQPT}}$ in Figs. 1(c)-(e). Using the smooth component of the Gutzwiller trace formula[56], we obtain a semiclassical approximation for the DOS[55]. This curve outlines the numerical data in Figs. 1(c)-(e).

Another consequence of the hyperbolic point is the onset of a positive Lyapunov exponent[53],

$$\lambda = 2K\xi,$$

(4)

The system described by Eq. (2) is regular, so the Lyapunov exponent for any initial condition is zero, except for the unstable point O[24, 57, 61].

Quantum dynamics: Instability.– The instability associated with the hyperbolic point is manifested in the quantum domain with the exponential growth of out-of-time-ordered correlators (OTOCs). We demonstrate this behavior using the FOTOC[53, 62],

$$F_{\text{otoc}}(t) = \sigma_p^2(t) + \sigma_q^2(t),$$

(5)

for initial coherent states with energies given by the points O, A-E marked in Fig. 2(a). State $|\Psi_A(0)\rangle$ has the lowest energy, followed by $|\Psi_B(0)\rangle$, $|\Psi_D(0)\rangle$, and $|\Psi_C(0)\rangle$. States $|\Psi_D(0)\rangle$ and $|\Psi_E(0)\rangle$ have the same high energy.

We compare the growth of $F_{\text{otoc}}(t)$ in Fig. 2(b) with the Husimi entropy

$$S_{H2}(t) = -\ln M_2(t),$$

(6)

in Fig. 2(c), where $M_2(t)$ is the integral of the square of the Husimi function[53]. Both quantities, $F_{\text{otoc}}(t)$ and $S_{H2}(t)$, measure how the evolving state spreads in the phase space. Snapshots of the evolution of the Husimi quasiprobability distributions for O, D, and E are also presented in the right three panels rows of Fig. 2 (more snapshots in[53]). The results are as follows.

(O): After the parabolic increase in $t$, that happens for very short times ($Kt < K\tau = (\sqrt{8}\xi)^{-1}$[53]), $F_{\text{otoc}}(O)(t)$ and $S_{H2}(O)(t)$ for the initial coherent state at the critical point, $|\Psi_O(0)\rangle$, grows exponentially [linearly] fast with a rate proportional to the classical Lyapunov exponent in Eq. (4), that is, $F_{\text{otoc}}(O)(t) \propto e^{\lambda t}$, $S_{H2}(O)(t) \propto \lambda t$. The snapshot of the Husimi function for a time as small as $Kt = 0.013$ (right side of Fig. 2) indicates that $|\Psi_O(0)\rangle$ is already very spread out in the phase space. Indeed, $F_{\text{otoc}}(O)(t)$ and $S_{H2}(O)(t)$ around this time reach the highest values among the states considered, as seen in Figs. 2(b)-(c).

The maximum value of $F_{\text{otoc}}(t)$ happens at the Ehrenfest time[53], given here by $T \sim \ln(\xi)/\lambda$. The fast scrambling of quantum information for $\tau < t < T$ is later followed by partial reconstructions of the initial distribution, as illustrated with the snapshot of the Husimi function at $Kt = 0.14$. In the absence of dissipation, the process of alternating spread and contraction persists for a long time[53].

(A): The initial coherent state $|\Psi_A(0)\rangle$ is very close a the center point, so the evolution is very slow and $F_{\text{otoc}}(A)(t)$ and $S_{H2}(A)(t)$ never reach large values.

(B) & (C): State $|\Psi_B(0)\rangle$ [|$\Psi_C(0)\rangle$] is slightly below [above] the ESQPT. Instead of the confinement around the center point imposed to the classical orbit B, quantum effects allow $|\Psi_B(t)\rangle$ to escape and evolve similarly to $|\Psi_C(t)\rangle$. The spread of the Husimi distributions for both states is comparable, reaching regions of the phase space with $+q$ and $-q$[53]. In addition, since B and C are in the vicinity of the unstable point O, quantum fluctuations trigger the exponential [linear] growth of $F_{\text{otoc}}(B,C)(t)$ and $S_{H2}(B,C)(t)$ seen in Fig. 2(b) [Fig. 2(c)]. This behavior is at odds with the classical limit, where the positive Lyapunov exponent emerges only at the hyperbolic point and not close to it.

(D) & (E): States $|\Psi_D(0)\rangle$ and $|\Psi_E(0)\rangle$ have the same high energy, but evolve differently. In terms of scrambling, $|\Psi_E(0)\rangle$ combines the best of both worlds, because in addition to high energy, it partially overlaps with the
separatrix (see the snapshot of the Husimi function at $t = 0$). As a result, $F_{otoc}^{(E)}(t)\ [S_{H2}^{(E)}(t)]$ in Fig. 2 (b) [Fig. 2 (c)] presents an exponential [linear] growth analogous to that seen for B and C, which is a quantum effect absent for $|\Psi_D(0)\rangle$. The spread of the Husimi distribution for $|\Psi_E(0)\rangle$ happens simultaneously inside and outside the separatrix, leading to complicated quantum interference effects, as those observed in the snapshot of the Husimi function at $Kt = 0.14$.

Quantum dynamics: Localization. – While the fastest scrambling happens for the initial coherent state $|\Psi_O(0)\rangle$, this state also presents the slowest decay of the survival probability,

$$S_p(t) = |\langle \Psi(0) | \Psi(t) \rangle|^2.$$  

The survival probability for all other initial coherent states, with energy above or below the ESQPT, decays faster than $S_p^{(O)}(t)$, as seen in Fig. 2 (d).

The apparent paradox of the fast spread of $|\Psi_O(t)\rangle$, as measured by $F_{otoc}^{(O)}(t)$ and $S_{H2}^{(O)}(t)$, and the slow decay of $S_p^{(O)}(t)$ is naturally resolved in view of the classical limit. The instability associated with the hyperbolic point O is the source of the exponentially fast spread of the variance of the phase-space distribution, but O is also a stationary point (the gradient of the Hamiltonian at this point is zero), so $|\Psi_O(0)\rangle$ is strongly localized in the eigenstate at the ESQPT [see Fig. 2 (k)]. In other words, the width of the energy distribution for $|\Psi_O(0)\rangle$, given by $\sqrt{2K\xi}$, is the smallest one. Close to the origin of the phase space, the evolution is dominated by the squeezing, $H_{q\rho} \approx \epsilon_2 (q^2 - \rho^2)$. This leads to the rapid stretching of $|\Psi_O(t)\rangle$, while part of the population remains for a long time in the vicinity of the origin. These two aspects of the dynamics become evident in the snapshot of the Husimi function for $|\Psi_O(t)\rangle$ at $Kt = 0.0075$. The small green ellipse in those panels indicates the size of the initial coherent state. One sees that the Husimi distribution for $|\Psi_O(t)\rangle$ stretches out without leaving the green ellipse.

Conclusions. – The closing of the spacing between pairs of adjacent energy levels (spectral kissing) recently observed in a squeezed Kerr oscillator [51] is a typical signature of an ESQPT. We advocate the use of this platform, where both frequency and time domain measurements can be done simultaneously, to observe dynamical behaviors induced by the presence of an ESQPT. We highlight the exponentially fast spread in the phase space of coherent states placed at the separatrix that marks the ESQPT. If the coherent state is also very close to the phase-space origin, it presents later revivals. If the state is far from the origin, it provides a good testbed for investigating the combined effects of fast scrambling and the subsequent interferences caused by the evolution inside and outside the separatrix.

We expect the squeezed Kerr oscillator to serve as a quantum simulator for nuclear, molecular, and condensed matter systems that present ESQPTs and related phenomena. As an example, we mention isomerizing systems, where the separation between neighboring energy levels decreases close to the isomerization barrier height [64, 65].

This research was supported by the NSF CCI grant (Award Number 2124511). T.L.M.L. was funded by the NSF grant No. DMR-1936006. F.P.B. was funded by the I+D+i project PID2019-104002GB-C21 (MCIN/AEI/10.13039/501100011033) and by the Consejería de Conocimiento, Investigación y Universidad, Junta de Andalucía and European Regional Development Fund (ERDF), ref. UHU-1262561. Computing resources supporting this work were partly provided by the CEA and Universidad de Huelva High Performance Computer (HPC@UHU) located in the Campus Uni-
versatorio el Carmen and funded by FEDER/MINECO project UNHU-15CE-2848. L.F.S. had support from the MPS Simons Foundation Award ID: 678586. J.C.C. and L.F.S. thank Jorge Hirsch and his group for various discussions on OTOCs and ESQPTs.


[63] D. Shepelyansky, Ehrenfest time and chaos, Scholarpedia 15, 55031 (2020)


[67] S. M. Girvin, Circuit qed: superconducting qubits coupled to microwave photons, Quantum machines: measurement, Cambridge University Press, 2018
Supplemental Material: Spectral kissing and its dynamical consequences in the squeezed Kerr-nonlinear oscillator

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Section I of this Supplemental Material contains details about the quantum and classical Hamiltonians of the squeezed Kerr-nonlinear oscillator and a discussion about the experimental parameters. Section II compares the plot for the excitation energies $E'$ as a function of the control parameter $\xi$ when both parities are considered with the plot for a single parity sector. Sections III, IV and V give the equations for the Husimi quasiprobability distribution, the density of states (DOS), and the Lyapunov exponent, respectively. Section VI provides additional figures for the quantum dynamics. In Sec. VII, we discuss how to derive the time interval for the initial quadratic behavior in $t$ of the survival probability, FOTOC, and $M_2(t)$. Section VIII shows the duration of the exponential growth of the FOTOC for coherent state $O$.

QUANTUM AND CLASSICAL HAMILTONIANS

In the same way that an LC circuit is the electrical analog of a mechanical harmonic oscillator, the Josephson junction is the electrical analog of a mechanical pendulum. The Hamiltonian of a single Josephson junction is

$$\hat{H} = \frac{1}{2C}\hat{Q}^2 - E_J \cos \left(\frac{2\pi}{\Phi_0} \hat{\Phi}\right),$$

where $C$ is the circuit’s capacitance, $E_J$ is the Josephson energy, $\hat{\Phi}$ is the phase circuit variable, and $\hat{Q}$ its charge, with $[\hat{\Phi}, \hat{Q}] = i\hbar$. This is the canonical commutation relation that describes quantum circuits and is analogous to the position-momentum relation in a mechanical system. The charge enters the Hamiltonian as
a quadratic kinetic energy and the circuit’s phase enters via the Josephson cosine potential, and is analogous to the projection of a constant gravitational field over the vertical as in a pendulum potential \cite{67}.

One defines the bosonic operators of the circuit as a convenient calculation tool. The annihilation operator for a superconducting circuit takes the form

\[ \hat{a} = \sqrt{\frac{1}{2\hbar Z}} (\hat{\Phi} + i Z\hat{Q}) \]  

(8)

where \( Z \) is the impedance of the circuit and \([\hat{a}, \hat{a}^\dagger] = 1\).

Alternatively one can write

\[ \hat{\Phi} = \Phi_{zpf} (\hat{a}^\dagger + \hat{a}) , \quad \hat{Q} = iQ_{zpf} (\hat{a}^\dagger - \hat{a}) , \]  

(9)

where \( \Phi_{zpf} = \sqrt{\hbar/2\omega C} = \sqrt{\hbar Z/2} \) is the zero point spread of the phase variable, \( \omega \) is the small oscillation frequency of the oscillator, and \( \Phi_{zpf} Q_{zpf} = \hbar/2 \). Insisting on the parallel with the mechanical oscillator, \( \Phi_{zpf} \) is the electrical analog to the ground state position uncertainty and \( Q_{zpf} \) corresponds to the ground state momentum uncertainty. The capacitance \( C \) then plays the role of the particle’s mass.

In the case of the SNAIL transmon used in Ref. \cite{51}, the Hamiltonian of the driven circuit, which is built by an arrangement of a few Josephson junctions, reads

\[ \frac{\dot{H}(t)}{\hbar} = \omega \hat{a}^\dagger \hat{a} + \sum_{m=3}^{\infty} \frac{g_m}{m} (\hat{a}^\dagger + \hat{a})^m - i\Omega_d (\hat{a} - \hat{a}^\dagger) \cos(\omega_d t) \]  

(10)

This is Eq. (1) in Ref. \cite{51}, where the \( g_n \)'s are the circuit nonlinearities and the drive is defined by its amplitude \( \Omega_d \) and its frequency \( \omega_d \), which is fixed at two times the small oscillation frequency of the oscillator to create resonant squeezing. Since nonlinearity is sourced by an arrangement of Josephson junctions in the SNAIL, the \( g_n \) coefficients are of order \( \Phi_{zpf}^{-2} \) \cite{68}. Additionally, the magnetic flux tuning of a SNAIL permits the tunability of the oscillator’s nonlinearities \cite{62}. In particular, one can tune the values of \( g_1(\Phi_B) \) and \( g_4(\Phi_B) \) rather accurately. For the sake of this discussion, we will approximate the impedance of the circuit as independent from the magnetic flux.

The static effective Hamiltonian describing the system in these conditions is given by

\[ \frac{\dot{H}}{\hbar} = -K\hat{a}^4 + \epsilon_2 (\hat{a}^2 + \hat{a}^\dagger)^2, \]  

(11)

where, from the microscopic theory introduced in \cite{52}, we can write the Kerr constant as

\[ K = -\frac{3K_B^2}{2} + 2\frac{g_0^2}{3\omega_d} \]  

and \( \epsilon_2 = g_2 \frac{4g_0}{3\omega_d} \) \cite{51}. This Hamiltonian is the quantum optical analog of a double-well potential \cite{69} and the number of levels inside the wells is given by \( N = \xi/\pi \) \cite{51}, where \( \xi = \epsilon_2/K \) is the control parameter.

We note that flux tuning a SNAIL circuit allows for a Kerr-free point \cite{70}, where the Kerr constant is null. We express this condition by writing \( K = \Phi_{zpf}^2 \kappa(\Phi_B) \), with \( \kappa(\Phi_B) \) a function crossing zero. In addition, the third-order nonlinearity responsible for the generation of squeezing remains essentially constant in the vicinity of the Kerr-free point, so we can write the scaling of the control parameter \( \xi \) as a function of the experimentally controllable variables,

\[ \xi = \frac{\epsilon_2}{K} \propto \frac{\Omega_d}{\Phi_{zpf}^2 \kappa(\Phi_B)}. \]  

(12)

With this expression, it is clear that the value of \( \xi \) can be increased in three different ways. One can (i) reduce the impedance of the circuit, thus reducing \( \Phi_{zpf} \), (ii) increase the microwave power of the squeezing drive \( \Omega_d \), or (iii) approach the Kerr-free point by in situ magnetic flux tuning \( \Phi_B \).

Classical limit

For large values of the control parameter, \( \xi \gg 1 \), the wells become very deep and the number of levels inside become macroscopic, so the quantum Hamiltonian in Eq. (11) exhibits properties comparable to the classical Hamiltonian. To derive the classical Hamiltonian for any depth of the wells, that is, to approach a continuous spectrum for a fixed and not necessarily large value of the control parameter, we introduce the parameter \( N_{eff} \), whose reciprocal is related with the size of the zero point fluctuations.

We write

\[ \dot{\hat{a}} = \sqrt{\frac{N_{eff}}{2}} (\hat{q} + i\hat{p}) , \]  

(13)

and

\[ [\hat{q}, \hat{p}] = \frac{i}{N_{eff}}, \]

so the classical limit can be reached by taking \( N_{eff} \to \infty \), since \( \hat{q} \to q, \hat{p} \to p \). This way, the quantum Hamiltonian,

\[ \frac{H_{qu}}{\hbar} = -\frac{K N_{eff}^2}{4} (\hat{q} - i\hat{p})^2 (\hat{q} + i\hat{p})^2 \]

\[ + \xi \frac{K N_{eff}}{2} [(\hat{q} - i\hat{p})^2 + (\hat{q} + i\hat{p})^2] , \]

(14)

leads to the classical Hamiltonian (with \( \hbar = 1 \)),

\[ H_{cl} = -\frac{K_{cl}}{4} (q^2 + p^2)^2 + K_{cl} \xi_{cl} (q^2 - p^2) , \]

(15)

where

\[ K = K_{cl}/N_{eff}^2 \]  

and \( \xi = \xi_{cl} N_{eff} \).
In the main text, we fixed

\[ N_{\text{eff}} = 1, \]

and used large values of \( \xi \). We also changed the sign of the Hamiltonians for convenience, so that we could say that \( E_0 \) in \( E' = E - E_0 \) is the ground state energy of \( \hat{H}_{\text{qu}} \), instead of its highest energy, and that the hyperbolic point is a local maximum, instead of a local minimum. Regardless of the sign convention, dissipation will bring the system to the attractors (stable nodes) in the bottom of the “wells”, which define unambiguously the ground state of the system.

The experimental system admits an approximate classical description if it is initialized in a coherent state and for as long as the Hamiltonian phase space surface produces only a linear force (a quadratic Hamiltonian) over the spread of the evolving state \[71\]. This means that the dynamics will be generated by the Poisson bracket, and used large values of \( \xi \). We also changed the sign of the Hamiltonians for convenience, so that we could say that \( E_0 \) in \( E' = E - E_0 \) is the ground state energy of \( \hat{H}_{\text{qu}} \), instead of its highest energy, and that the hyperbolic point is a local maximum, instead of a local minimum. Regardless of the sign convention, dissipation will bring the system to the attractors (stable nodes) in the bottom of the “wells”, which define unambiguously the ground state of the system.

The experimental system admits an approximate classical description if it is initialized in a coherent state and for as long as the Hamiltonian phase space surface produces only a linear force (a quadratic Hamiltonian) over the spread of the evolving state \[71\]. This means that the dynamics will be generated by the Poisson bracket, since the Moyal corrections can be neglected, and no phase space interference effects will develop. This can be achieved by reducing the fluctuations of the coherent state (increasing its “mass”, \( \Phi_{\text{zpf}} \rightarrow 0 \)), or by making a comparatively large double-well system \( (\Omega_d/\kappa B) \rightarrow \infty \). Note that reducing \( \Phi_{\text{zpf}} \) comes at the price of increasing the spread in the momentum coordinate. In a Hamiltonian with quadratic kinetic energy, like Eq. \[10\], this comes at a minimal cost, since no nonlinearity is experienced along the momentum (charge) axis, and the Moyal corrections remain small. Note, however, that in the presence of a Kerr nonlinearity, the Hamiltonian has a nonlinear dependence on the momentum coordinate and the classical correspondence needs to be treated carefully. This justifies taking the classical limit as a system of increasing size, \( \xi \gg 1 \), as in the main text, which can be achieved independently of the value of zero point fluctuations.

In the absence of dissipation, this classical Hamiltonian approximation breaks at sufficiently long times for most initial conditions. In turn, small amounts of dissipation enforce the classical dynamics \[51, 72, 73\]. As in \[72, 74\], the system discussed here is not chaotic, but since for a state initialized near the ESQPT, the evolution can be approximated by a quadratic Hamiltonian (squeezing, \( \epsilon_2 \gg K \)), the exponential instability is a property of both the quantum and the classical models. The evolution can be approximated as classical until the phase space distribution folds on the quartic energy wall and develops phase interferences, such as those seen in the last snapshot of the last row of Fig. 2 in the main text \[75\]. This quantum-classical divergence will be regularized in a timescale set by dissipation. The possibility to experimentally explore the quantum-classical correspondence in the squeezed Kerr oscillator will be communicated elsewhere.

**CLUSTERING OF EIGENVALUES**

Figure 3 (a) is identical to Fig. 1 (b) in the main text. Figure 3 (b) is the same as Figure 3 (a), but displayed for a single parity sector with the purpose of making it evident that the clustering of the eigenvalues at \( E'_{\text{ESQPT}} \) happens in a single sector as well. The size \( N \) of the truncated Hilbert space here and everywhere in this work is chosen to guarantee the convergence of the levels analyzed.

**HUSIMI QUASIPROBABILITY DISTRIBUTION**

For an eigenstate written in the basis of the Glauber coherent states,

\[ |\alpha⟩ = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{N_{\text{eff}}} \frac{\alpha^n}{\sqrt{n!}} |n⟩, \]

where \( \hat{a}|\alpha⟩ = \alpha|\alpha⟩ \), \( N \) is the truncation of the Hilbert space, and

\[ \alpha = \sqrt{\frac{1}{2}} (q + ip) \]

for \( N_{\text{eff}} = 1 \), the Husimi quasiprobability distribution is given by \[76\]

\[ Q^e(q,p) = \frac{1}{2\pi} \left| \sum_{n=0}^{N_{\text{eff}}} C_n e^{-\frac{n^2}{2} + n} (q - ip)^n \sqrt{2^n n!} \right|^2. \]

**DENSITY OF STATES**

We can use the lowest-order term of the Gutzwiller trace formula \[56\] to obtain a semiclassical approximation for the DOS,

\[ \nu(\mathcal{E}) = \frac{1}{2\pi} \int dp dq \delta(H_{cl} - \mathcal{E}), \]
where $H_{cl}$ is given in Eq. [15] [Eq. (2) of the main text]. To evaluate the previous integral, we use the general property of the Dirac delta,

$$\int_{\mathbb{R}^n} f(x)\delta(g(x))dx = \int_{g^{-1}(0)} \frac{f(x)}{|\nabla g|} d\sigma(x),$$  \hspace{1cm} (19)

where the integral on the right is over $g^{-1}(0)$ and the $(n-1)$-dimensional surface defined by $g(x) = 0$. Employing the property of the Dirac delta in the Gutzwiller formula, we have

$$\nu(\mathcal{E}) = \frac{1}{2\pi} \int_{q \in \Omega_\mathcal{E}} dq \frac{1}{2\sqrt{(2\sqrt{K_{cl}} u(\mathcal{E}) - (\lambda + K_{cl} q^2)) u(\mathcal{E})}},$$  \hspace{1cm} (20)

where $u(\mathcal{E}) = \mathcal{E} - \mathcal{E}_{\text{min}} + \lambda q^2$, $\lambda = 2K_{cl} \xi_{cl}$, and $\Omega_\mathcal{E}$ is the set of values of $q$ for which there is at least one solution of the equation $H_{cl}(q, p) = \mathcal{E}$.

**LYAPUNOV EXPONENT**

The linear analysis around critical points gives us information about the qualitative behavior close to those points. In particular, for the Hamiltonian in Eq. [15], the linearized Hamilton equations around a critical point $\{ q_c, p_c \}$ satisfy

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = 2K_{cl} q_c p_c \begin{bmatrix} 2K_{cl} \xi_{cl} + K_{cl}(q_c^2 + 3p_c^2) & 2K_{cl} + K_{cl}(q_c^2 + 3p_c^2) \\ -2K_{cl} q_c p_c & -2K_{cl} q_c p_c \end{bmatrix} \begin{bmatrix} q - q_c \\ p - p_c \end{bmatrix}.$$

In the equation above, cl stands for “classical” and c for “critical”. The stability or instability around $\{ q_c, p_c \}$ is given by the eigenvalues $\alpha_i$ of the matrix constructed by the linear system. If the eigenvalues of the matrix are real, then the Lyapunov exponent is equal to $\max(\alpha_i)$. Specifically, for the critical point $\{ q_c, p_c \} = \{ 0, 0 \}$, which is a hyperbolic point, the Lyapunov exponent is given by

$$\lambda = 2K_{cl} \xi_{cl}. \hspace{1cm} (21)$$

**QUANTUM DYNAMICS**

The 6 initial coherent states that we consider are obtained by using in Eq. [16] the values of $p$ and $q$ specified in Table I. These are the points marked in Fig. 2 (a) of the main text.

**FOTOC**

The FOTOC $F_{\text{OTOC}}$ is derived from the commutator $\langle [\hat{W}(t), \hat{V}(0)]^2 \rangle$ that defines the OTOC $\Gamma$, when the operator $\hat{V} = |\Psi(0)\rangle \langle \Psi(0)|$ is the projection operator on the initial state $|\Psi(0)\rangle$, and $\hat{W} = e^{i\delta \phi \hat{G}}$, where $\delta \phi$ is a small perturbation and $\hat{G}$ is a Hermitian operator. In the perturbative limit, $\delta \phi \ll 0$, the FOTOC is the variance $\sigma_\phi^2(t) = \langle \hat{G}^2(t) \rangle - \langle \hat{G}(t) \rangle^2$. In this work, we consider

$$F_{\text{OTOC}}(t) = \sigma_\phi^2(t) + \sigma_\phi^2(t). \hspace{1cm} (22)$$

**Integral of the square of the Husimi function**

One can quantify how an initial coherent state spreads in the phase space by computing the integral of the square of the Husimi quasiprobability distribution,

$$M_2^{\Psi}(t) = \frac{1}{2\pi} \int dq dp |Q^{\Psi}(t)(q, p)|^2.$$

where $N_{\text{eff}} = 1$ and $Q^{\Psi}(q, p)$ is given in Eq. [17], but now instead of an eigenstate $|\psi\rangle$, as written in that equation, we use the evolved state $|\Psi(t)\rangle$.

By writing the evolved state in the Fock basis, $|\Psi(t)\rangle = \sum_n C_n(t)|n\rangle$, one can solve the integrals exactly and obtain

$$M_2^{\Psi}(t) = \frac{1}{\pi} \sum_{n_1, n_2, m_1, m_2} \frac{C_{n_1}(t) C^*_{n_2}(t) C_{m_1}(t) C^*_{m_2}(t)}{\sqrt{n_1! n_2! m_1! m_2!}} \times \int d^2\alpha e^{-2|\alpha|^2} \epsilon^{* n_1 + n_2 + m_1 + m_2} \delta_{n_1 + m_1, n_2 + m_2} \hspace{1cm} (24)$$

**Snapshots of the evolution of the Husimi quasiprobability distribution**

This subsection presents various snapshots of the evolution of the Husimi functions for the 6 initial coherent states investigated, including again the same cases presented in the main text for $|\Psi_O(0)\rangle$, $|\Psi_D(0)\rangle$, and $|\Psi_E(0)\rangle$. The main features are summarized below.

The three rows of panels on the top left of Fig. [3] present snapshots of the Husimi function for four instants of time.

<table>
<thead>
<tr>
<th>Point</th>
<th>$q$</th>
<th>$p$</th>
<th>$\mathcal{E}[10^4]/K_{cl}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>O</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>A</td>
<td>16.9143</td>
<td>0</td>
<td>-3.1034</td>
</tr>
<tr>
<td>B</td>
<td>1.2533</td>
<td>0</td>
<td>-0.0282</td>
</tr>
<tr>
<td>C</td>
<td>0</td>
<td>1.2506</td>
<td>0.0282</td>
</tr>
<tr>
<td>D</td>
<td>0</td>
<td>8.4443</td>
<td>1.4106</td>
</tr>
<tr>
<td>E</td>
<td>28.1302</td>
<td>0</td>
<td>1.4106</td>
</tr>
</tbody>
</table>

**TABLE I.** The table indicates the points in phase space that determine the initial coherent states used for the quantum dynamics; $\mathcal{E}$ is the classical energy of each point for $\xi_{cl} = 180$. 

for the initial coherent states $|\Psi_A(0)\rangle$ (first row), $|\Psi_B(0)\rangle$ (second row), and $|\Psi_C(0)\rangle$ (third row). State A has very low energy and thus exhibits a very limited spread around its initial region in the phase space. At $Kt = 0.02, 0.13$, the distribution gets mostly out of the green ellipse that determines the initial state, so the value of $S_p^{(A)}(t)$ should become very small.

In contrast with $|\Psi_A(t)\rangle$, the Husimi distributions for $|\Psi_B(t)\rangle$ and $|\Psi_C(t)\rangle$ get squeezed, but do not fully leave the green ellipse. These two states present evolutions similar to the coherent state $|\psi_O(t)\rangle$, since the two also start close to the critical point at the origin of the phase space. As mentioned in the main text, the fact that $|\Psi_B(t)\rangle$ evolves towards the region with negative values of $q$ is a quantum effect. The classical point B has a positive value of $q$ and is inside the separatrix, so classically its orbit never reaches values of $q < 0$.

The various snapshots of the Husimi functions for $|\Psi_A(t)\rangle$ (top right), $|\Psi_D(t)\rangle$ (bottom left), and $|\Psi_E(t)\rangle$ (bottom right) complement those displayed in the main text. The panels for $|\Psi_O(t)\rangle$ make evident the fast spread of this state, and also the subsequent alternating spread and contraction of its Husimi function.

State E also spreads fast, because it is placed on the separatrix, although far from the origin. Just as for B and C, its exponential behavior is a quantum effect.

Part of the quantum evolution of the coherent state E happens inside the separatrix and part of it is outside, creating two spreading fronts, as visible from the snapshots at $Kt = 0.027, 0.037, 0.04$, and 0.05. These different paths generate a complicated pattern of interferences, as shown for $Kt = 0.13$ and 0.14. Quantum interferences also appear for the initial coherent state $|\Psi_D(0)\rangle$. This state has high energy equal to state E, but since $|\Psi_D(0)\rangle$ starts far from the separatrix, the interferences take longer to develop.

### Quadratic Behavior in Time

At very short times, the survival probability, FOTOC, and $M_2(t)$ present a quadratic behavior in time. The time interval for this behavior is derived by doing a Taylor expansion of the propagator $U(t) = e^{-iHt}$ as shown below.

#### Survival Probability

At short times, the survival probability can be written as

$$S_p(t) = \left|\langle\Psi(0)|e^{-iHt}|\Psi(0)\rangle\right|^2 \approx \left|\left\langle\Psi(0)| (1 - iHt - \frac{H^2t^2}{2})|\Psi(0)\rangle\right\rangle\right|^2 = 1 - t^2 \left[\langle\Psi(0)|H|^2|\Psi(0)\rangle - \langle\Psi(0)|H|\Psi(0)\rangle^2\right] = 1 - \Gamma^2 t^2,$$

where $\Gamma^2$ is the variance of the energy distribution of the initial state written in the energy eigenbasis, that is

$$\Gamma^2 = \sum_k |C_k(0)|^2 (E_k - E_0)^2,$$

where $H|E_k\rangle = E_k|E_k\rangle$, $E_0 = \langle\Psi(0)|H|\Psi(0)\rangle$, and $C_k(0) = \langle E_k|\Psi(0)\rangle$.

Using the Fock basis $|n\rangle$ to write $\Gamma_O^2$ for the initial coherent state $O$, we have that

$$\Gamma_O^2 = \sum_n \langle 0|H|n\rangle \langle n|H|0\rangle - \langle 0|H|0\rangle^2 = \sum_{n \neq 0} |\langle n|H|0\rangle|^2,$$

therefore,

$$S_p^O(t) \approx 1 - 2\xi^2 K^2 t^2.$$  \hspace{1cm} (25)

This implies that the survival probability for the state O decays quadratically for

$$Kt < \frac{1}{\sqrt{2\xi}}.$$  \hspace{1cm} (26)

The derivation of $\Gamma^2$ for the other coherent states is analogous. As evident from Fig. 2 (d) in the main text, the state $|\Psi_O(0)\rangle$ has the smallest variance $\Gamma_O^2$, which happens because the corresponding classical point $O$ is a stationary point. The gradient and Laplacian of the Hamiltonian vanish at $O$, so the initial diffusion constant for the Glauber coherent state $|\Psi_O(0)\rangle$ is the smallest one.

In Fig. 5 we show the energy distributions of the coherent states O, D, and E. The width of the distribution for O is significantly narrower than for the other two states, as anticipated from the dynamics.

Another feature observed in Fig. 6 is the difference in the widths of the energy distributions for coherent states D and E. Even though both initial states have the same energy, coherent state $|\Psi_E(0)\rangle$ is more spread out than $|\Psi_D(0)\rangle$, which explains why the survival probability $S_p^E(t)$ decays faster than $S_p^D(t)$, as seen in the Fig. 2 (d) of the main text.
FIG. 4. Snapshots of the Husimi quasiprobability distributions for the 6 initial coherent states investigated, as indicated in the titles; $\xi = \xi_{cl} = 180$. On the top left, the snapshots in the first row of panels are for state A, in the second row they are for B, and the third row for C.

FIG. 5. Energy distribution of the coherent states O, D, and E, as indicated in the panels; $\xi = 180$, $\hbar = 1$.

**FOTOC**

The same procedure can be extended to the FOTOC, where one now needs to compute

$$\langle \Psi(0) \left| \left[ 1 + iHt - \frac{H^2t^2}{2} \right] \hat{A} \left[ 1 - iHt - \frac{H^2t^2}{2} \right] \right| \Psi(0) \rangle$$

up to $O(t^2)$ for $\hat{A} = \hat{p}$, $\hat{A} = \hat{p}^2$, $\hat{A} = \hat{q}$, and $\hat{A} = \hat{q}^2$.

For the coherent state O, we find that

$$F^{(O)}_{FOTOC} \approx 1 + 8\xi^2 K^2 t^2,$$

so its quadratic behavior holds for

$$Kt < \frac{1}{\sqrt{8} \xi}.$$
Integral of the square of the Husimi function

To determine the duration of the quadratic behavior of $M_2(t)$, one needs to do the Taylor expansion for each component $C_n(t)$ in Eq. (24), which becomes a tedious exercise even for the coherent state $|O\rangle$. This timescale should again be dependent on the value of the control parameter, and we verify numerically that

$$K t < \frac{1}{\xi}. \quad (29)$$

is an upper bound.

EXPOENTIAL GROWTH OF THE FOTOC

The exponential growth of the FOTOC for the coherent state $|\Psi_O(0)\rangle$ holds up to the Ehrenfest time $T$, which in our case is given by

$$K T \sim -0.0027 + \ln(\xi)/(2\xi). \quad (30)$$

In Fig. 6, we show numerical results for $K T$ as a function of $\xi$, and we find very good agreement with the expression in Eq. (30).

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