An integrated approach to ladder and shift operators for the Morse oscillator, radial Coulomb and radial oscillator potentials

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Ian L Cooper

Department of Chemistry, The University, Newcastle upon Tyne NE1 7RU, UK

Abstract. The Morse oscillator, radial Coulomb and radial harmonic oscillator problems can be solved exactly using a variety of algebraic methods. These problems correspond to different realizations of the $so(2,1)$ algebra and a comparison of the generators of the algebra may be used to identify mappings between each pair of systems. The resultant transition operators act as ladder, or energy changing, operators in the cases of the Coulomb and harmonic oscillator potentials, whereas they act as shift operators, acting at constant energy, in the case of the Morse potential. This is a consequence of the $so(2,1)$ dynamical symmetry, whereby the Morse Hamiltonian is expressible solely in terms of the Casimir operator of the algebra. An alternative algebraic approach, the use of the method of supersymmetric quantum mechanics, or factorization, produces in each case a set of shift operators. Relations between the various ladder and shift operators may be identified by means of the appropriate mappings, and these results can be generalized so as to relate the one dimensional Morse oscillator to the radial Coulomb and radial harmonic oscillator potentials involving an arbitrary number of angular dimensions.

1. Introduction

The application of algebraic methods to quantum mechanical problems has seen rapid expansion in recent years. The factorization method (Schrödinger 1940, Infeld and Hull 1951) has received renewed attention following the development of supersymmetric quantum mechanics (Witten 1981, Sukumar 1985), and it has been demonstrated that the two approaches are indeed equivalent (Alves and Drigho Filho 1988, Montemayor and Salem 1989). Systems displaying dynamical symmetry can also be treated with algebraic techniques (Arima and Iachello 1974, Perelomov 1985) and the potential group approach has been applied (Alhassid *et al* 1983, Wu and Alhassid 1990, Englefield and Quesne 1991) to the case of the Morse potential, thereby relating states of the same energy but belonging to different potential strengths (i.e corresponding to displaced Morse potentials). This is also related to the concept of shape invariance (Gendenshtein 1983), and it has been demonstrated (Alves and Drigho Filho 1988) that the set of shape invariant potentials is essentially the same as that obtained by the factorization approach.

We shall consider here three important exactly solvable potentials, namely the radial Coulomb, Morse oscillator and radial harmonic oscillator potentials, which have been the subject of considerable interest in the development of algebraic methods to exactly solvable problems. These problems have been shown to correspond to

different realizations of $so(2,1)$ symmetry (Čížek and Paldus 1977, Berrondo and Palma 1980, Wu and Alhassid 1990) and have all been treated by the methods of factorization or supersymmetric quantum mechanics (Haymaker and Rau 1986, Dutt *et al* 1987, Ding 1987, Drigho Filho 1988). In the present paper, these algebraic methods will be applied to the above three solvable potentials within the context of a common framework in order to construct the set of transition operators which connect adjacent energy eigenstates, and the common $so(2,1)$ symmetry will be exploited to generate explicit mappings between the transition operators for each pair of potentials. Operators connecting states of the same energy will be termed shift operators to distinguish them from ladder operators connecting states of different energy. These ladder and shift operators are of considerable value in the algebraic determination of matrix elements involving exact eigenstates. The set of ladder and shift operators for the Coulomb problem will be shown to be derivable from a combination of $so(2,1)$ symmetry and the method of supersymmetric quantum mechanics, in contrast to the Morse potential, for which the two approaches lead to the *same* set of shift operators connecting Morse potentials displaced in both well depth and equilibrium position. This can be traced to the different structures of the two problems (Wehrhahn and Cooper 1992), in particular to the dynamical symmetry of the Morse problem whereby the Hamiltonian is related to the Casimir of the algebra. In the case of the radial harmonic oscillator, application of $so(2,1)$ symmetry generates ladder operators at a constant value of angular momentum quantum number, whereas application of the method of supersymmetric quantum mechanics leads to shift operators between eigenstates which are forced into degeneracy through displacements in the zero of energy. Such displacements are permissible in this particular problem since there is no dissasociation or ionization limit to act as a constraint. Ladder operators for the Morse oscillator and true (constant energy) shift operators for the radial harmonic oscillator can be generated from the Coulomb shift operators by the use of the appropriate mappings.

The plan of the current paper is as follows. In the next section, we discuss a unified algebraic approach to bound states of the three problems based on $so(2,1)$ symmetry. The relevant transition operators are identified, together with mappings between the bound states of each pair of problems. The following section discusses each problem from the point of view of supersymmetry or factorization, and various connections between the resultant shift operators and the transition operators arising from $so(2,1)$ symmetry are identified. The mappings between each pair of problems are then used to generate the remaining ladder and shift operators in each case and the inter-relations are displayed explicitly. Generalization of these results to an arbitrary number of angular dimensions in the case of the Coulomb and harmonic oscillator problems are then straightforward. The paper ends with some concluding remarks.

2. $so(2,1)$ algebraic treatment

Algebraic approaches to the Coulomb problem are well established (see, for example, Englefield 1972) as are those for the Morse oscillator (Huffaker and Dwivedi 1975, Berrondo and Palma 1980). Here, we shall present a simple unified approach to the radial Coulomb, Morse oscillator and radial harmonic oscillator problems using $so(2,1)$ symmetry, following the treatment of Čížek and Paldus (Čížek and Paldus

1977, Berrondo and Palma 1980). This particular approach, which was developed for bound state problems, demonstrates that the above three systems correspond to different realizations of the algebra $so(2,1)$. Although the method can be shown to generate the eigenvalue spectrum directly, we shall assume for convenience that the eigenvalues are known in each case.

2.1. Coulomb problem

In atomic units, the radial Schrödinger equation for a one electron atom with nuclear charge Z becomes

$$\left[-\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} - \frac{2Z}{r} + \frac{Z^2}{n^2}\right]\psi_{v,l}(r) = 0 \tag{2.1}$$

where l is the angular momentum quantum number, $n(= v + l + 1)$ is the principal quantum number, and v (used in preference to n_r in order to facilitate comparisons among the various systems) is a quantum number which denotes the number of radial nodes.

We introduce the n-dependent variable ρ by the relation

$$\rho = \frac{Z}{n}r \tag{2.2}$$

and equation (2.1), after multiplication by ρ and rearrangement, gives

$$\left[-\rho\frac{d^2}{d\rho^2} + \frac{l(l+1)}{\rho} + \rho\right]\psi_{v,l} = 2n\psi_{v,l}. \tag{2.3}$$

We now define the operators

$$W_1 = \rho \tag{2.4}$$

$$W_3 = -\rho\frac{d^2}{d\rho^2} + \frac{l(l+1)}{\rho} \tag{2.5}$$

with commutator

$$[W_1, W_3] = 2\rho\frac{d}{d\rho} = 2iW_2 \tag{2.6}$$

where W_2 is given by

$$W_2 = -i\rho\frac{d}{d\rho}. \tag{2.7}$$

The set of operators

$$\begin{aligned}
T_1 &= \tfrac{1}{2}(W_3 - W_1) \\
T_2 &= W_2 \\
T_3 &= \tfrac{1}{2}(W_3 + W_1)
\end{aligned} \tag{2.8}$$

have commutation relations characteristic of the algebra $so(2,1)$ (Englefield 1972, Wybourne 1974)

$$[T_1, T_2] = -iT_3$$
$$[T_2, T_3] = iT_1 \qquad (2.9)$$
$$[T_3, T_1] = iT_2.$$

Equation (2.2) then becomes an eigenvalue equation for T_3, with eigenvalue n, namely

$$T_3\psi_{v,l} = n\psi_{v,l} \equiv (v+l+1)\psi_{v,l}. \qquad (2.10)$$

The Casimir operator of the algebra $so(2,1)$ is given by

$$C_2 = T_3^2 - T_1^2 - T_2^2 = W_1 W_3 - W_2(W_2+i) = l(l+1) \qquad (2.11)$$

so that

$$C_2\psi_{v,l} = l(l+1)\psi_{v,l}. \qquad (2.12)$$

Since the eigenvalues of the Casimir operator of the algebra are characterized by the value of the angular momentum quantum number, the irreducible representations are characterized by l. From equation (2.10) we note that the eigenvalues of the Casimir, T_3, of the subalgebra $so(2)$ correspond to the principal quantum number and hence determine the energy.

We may construct transition operators by the presecription

$$T_\pm = T_1 \pm iT_2 = \tfrac{1}{2}(W_3 - W_1) \pm iW_2. \qquad (2.13)$$

Hence

$$T_\pm = \pm\rho\frac{d}{d\rho} + \frac{1}{2}\rho\left[-\frac{d^2}{d\rho^2} + \frac{l(l+1)}{\rho^2} - 1\right]$$
$$\equiv \pm\rho\frac{d}{d\rho} - \rho + n + \frac{1}{2}\rho\left[-\frac{d^2}{d\rho^2} + \frac{l(l+1)}{\rho^2} - \frac{2n}{\rho} + 1\right]. \qquad (2.14)$$

Provided that the operators $T_\pm$ are assumed to act on eigenstate $\psi_{v,l}$, with $n = v + l + 1$, then equation (2.14), in conjunction with equation (2.3), becomes

$$T_\pm = \pm\rho\frac{d}{d\rho} - \rho + n. \qquad (2.15)$$

Note that the operators depend both explicitly and implicitly (via $\rho = Zr/n$) on the value of n and act on an eigenstate which is characterized by this value of n.

From the commutation relations (2.9), we find

$$T_3 T_\pm = T_\pm(T_3 \pm 1) \qquad (2.16)$$

so that

$$T_3(T_\pm \psi_{v,l}) = (n \pm 1)(T_\pm \psi_{v,l}). \qquad (2.17)$$

Thus $T_{\pm}\psi_{v,l} \propto \psi_{v\pm 1,l}$, and the constants of proportionality may be determined as follows:

Since

$$T_{\pm}T_{\mp} \equiv T_3(T_3 \mp 1) - C_2 = [n(n \mp 1) - l(l+1)] \tag{2.18}$$

we have

$$T_+T_-\psi_{v,l} = [n(n-1) - l(l+1)]\psi_{v,l} \tag{2.19}$$

so that, choosing a phase factor of unity

$$T_-\psi_{v,l} = [n(n-1) - l(l+1)]^{1/2}\psi_{v-1,l}. \tag{2.20}$$

Similarly, from the relation

$$T_-T_+\psi_{v,l} = [n(n+1) - l(l+1)]\psi_{v,l} \tag{2.21}$$

we have

$$T_+\psi_{v,l} = [n(n+1) - l(l+1)]^{1/2}\psi_{v+1,l}. \tag{2.22}$$

These represent ladder operators for the Coulomb problem, changing the energy within an irreducible representation of $so(2,1)$ i.e. at constant value of l. However, care must be taken with respect to the dependence on ρ. Specifically, these operators convert a function of $\rho = Zr/n$ into a function of $\rho = Zr/(n \pm 1)$. Also, since we are dealing with radial wavefunctions, the correctly normalized wavefunctions should be multiplied by the radial coordinate r, which has to be converted to the n-dependent coordinate ρ. This requires that we multiply $\psi_{v,l}$ by n/Z, and $\psi_{v\pm 1,l}$ by $(n \pm 1)/Z$, noting that the term in Z will cancel out. Hence, denoting these Coulomb ladder operators by $L_{\pm}$, we have

$$L_{\pm} = \pm\rho\frac{\mathrm{d}}{\mathrm{d}\rho} - \rho + n \tag{2.23}$$

such that

$$L_{\pm}\psi_{v,l} = \left(\frac{n \pm 1}{n}\right)[n(n \pm 1) - l(l+1)]^{1/2}\psi_{v\pm 1,l} \tag{2.24}$$

where $n = v + l + 1$. Note that $L_-\psi_{0,l} \equiv 0$ as required, and this result may be used to derive the ground-state eigenfunctions for specified values of l by solution of the corresponding first order differential equations.

2.2. Radial harmonic oscillator

The radial Schrödinger equation for the harmonic oscillator is

$$\left[-\frac{\hbar^2}{2\mu}\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{l(l+1)\hbar^2}{2\mu r^2} + \frac{1}{2}\mu\omega^2 r^2\right]\psi_{v,l} = \hbar\omega(2v + l + 3/2)\psi_{v,l} \tag{2.25}$$

where μ is the reduced mass and ω is the angular vibrational frequency of the oscillator, l is the angular momentum quantum number and v is the vibrational quantum number. Transforming to the dimensionless variable $\xi = r/a$ where $a = (\hbar/\mu\omega)^{1/2}$ allows us to rewrite equation (2.25) as

$$\left[-\frac{1}{2}\frac{d^2}{d\xi^2} + \frac{l(l+1)}{2\xi^2} + \frac{1}{2}\xi^2 - (2v + l + 3/2)\right]\psi_{v,l} = 0. \tag{2.26}$$

Following the procedure in subsection 2.1 above, we define the generators

$$W_1 = \frac{1}{2}\xi^2 \tag{2.27}$$

$$W_3 = \frac{1}{2}\left[-\frac{d^2}{d\xi^2} + \frac{l(l+1)}{\xi^2}\right] \tag{2.28}$$

with the commutator

$$[W_1, W_3] = \xi\frac{d}{d\xi} + \frac{1}{2} = 2iW_2 \tag{2.29}$$

where

$$W_2 = -\frac{i}{2}\left(\xi\frac{d}{d\xi} + \frac{1}{2}\right). \tag{2.30}$$

Defining the operators T_i ($i = 1, 2, 3$) as in equation (2.8) with commutation relations given in equation (2.9), yields, in this case, for the operators T_3 and C_2 the relations

$$T_3\psi_{v,l} = \frac{1}{2}(2v + l + 3/2)\psi_{v,l} \tag{2.31}$$

and

$$C_2\psi_{v,l} = (T_3^2 - T_1^2 - T_2^2)\psi_{v,l} = \frac{1}{4}[l(l+1) - 3/4]\psi_{v,l}. \tag{2.32}$$

As in the Coulomb case, the Casimir operator of the algebra $so(2,1)$ determines the angular momentum quantum number and the Casimir operator of the sub-algebra $so(2)$ determines the energy of the oscillator. In this case, the transition operators $T_\pm$ have the form, when operating on state $\psi_{v,l}$

$$T_\pm = \frac{1}{2}\left(\pm\xi\frac{d}{d\xi} \pm \frac{1}{2} - \xi^2 + 2v + l + \frac{3}{2}\right) \tag{2.33}$$

where we have dropped a term which is annihilated when operating on the state $\psi_{v,l}$. Since the commutation relations are unchanged we still have

$$T_\mp T_\pm = T_3(T_3 \pm 1) - C_2 \tag{2.34}$$

so that

$$T_\pm\psi_{v,l} = [T_3(T_3 \pm 1) - C_2]^{1/2}\psi_{v\pm1,l}. \tag{2.35}$$

The effect of T_3 and C_2 in the present instance are given by equations (2.31) and (2.32) respectively.

Thus, as in the Coulomb case, we generate ladder operators $L_\pm$ which act to change the energy at constant value of the angular momentum quantum number. Explicitly, these ladder operators have the form

$$L_\pm = \frac{1}{2}\left(\pm\xi\frac{\mathrm{d}}{\mathrm{d}\xi} \pm \frac{1}{2} - \xi^2 + 2v + l + \frac{3}{2}\right) \qquad (2.36)$$

such that, from equations (2.31) and (2.32)

$$L_\pm\psi_{v,l} = 1/2[(2v + l + 3/2)(2v + l + 3/2 \pm 2) - l(l+1) + 3/4]^{1/2}\psi_{v\pm1,l}. \quad (2.37)$$

Note that, as in the Coulomb case, the precise form of these ladder operators depends on the precise eigenstate upon which they act.

2.3. Morse oscillator

We now consider the $so(2,1)$ algebraic approach to the Morse oscillator (Berrondo and Palma 1980). The Morse Hamiltonian is given by

$$H = -\frac{\hbar^2}{2\mu}\frac{\mathrm{d}^2}{\mathrm{d}R^2} + D_e(1 - e^{-\alpha(R-R_e)})^2 - D_e \qquad (2.38)$$

where D_e is the well depth, μ is the reduced mass, R_e is the equilibrium separation and α is the range parameter. Transforming to the dimensionless variable $y = \alpha(R-R_e)$ and defining the well depth parameter λ via the relation $\lambda^2 = 2\mu D_e/\alpha^2\hbar^2$, we obtain the reduced Hamiltonian (in units of $\alpha^2\hbar^2/2\mu$)

$$H = -\frac{\mathrm{d}^2}{\mathrm{d}y^2} + \lambda^2(1 - e^{-y})^2 - \lambda^2. \qquad (2.39)$$

The eigenvalues of this Hamiltonian are well known (Morse 1929)

$$E_v = -(\lambda - v - 1/2)^2 \qquad (2.40)$$

where v, the vibrational quantum number, takes the values $v = 0, 1, 2, \ldots, v_{\max}$ with $v_{\max} = \mathrm{Int}(\lambda - 1/2)$, such that the number of bound vibrational levels is equal to Int $(\lambda + 1/2)$ at specified value of well depth parameter λ. Using the above eigenvalue expression, we can rewrite the Schrödinger equation for the Morse oscillator in the form

$$\left[-e^y\frac{\mathrm{d}^2}{\mathrm{d}y^2} + (\lambda - v - 1/2)^2 e^y + \lambda^2 e^{-y}\right]\psi_{v,\lambda} = 2\lambda^2\psi_{v,\lambda}. \qquad (2.41)$$

If we now define a new variable x according to the prescription

$$e^{-x} = \lambda e^{-y} \qquad \text{or} \qquad x = y - \ln\lambda \qquad (2.42)$$

then equation (2.41) takes the form

$$\left[-e^x\frac{\mathrm{d}^2}{\mathrm{d}x^2} + (\lambda - v - 1/2)^2 e^x + e^{-x}\right]\psi_{v,\lambda} = 2\lambda\psi_{v,\lambda}. \qquad (2.43)$$

Following the procedures described previously for the radial Coulomb and radial harmonic oscillator problems, we define the operators

$$W_1 = e^{-x} \tag{2.44}$$

$$W_3 = -e^x \frac{d^2}{dx^2} + (\lambda - v - 1/2)^2 e^x \tag{2.45}$$

with the commutator

$$[W_1, W_3] = 1 - 2\frac{d}{dx} = 2iW_2 \tag{2.46}$$

where

$$W_2 = i\left(\frac{d}{dx} - \frac{1}{2}\right). \tag{2.47}$$

Again introducing the operators T_i $(i = 1, 2, 3)$ as in equation (2.8), with commutators given by equation (2.9), equation (2.43) becomes

$$T_3 \psi_{v,\lambda} = \lambda \psi_{v,\lambda} \tag{2.48}$$

whereas, from the Casimir $C_2 (= T_3^2 - T_1^2 - T_2^2)$ of the algebra, we have

$$C_2 \psi_{v,\lambda} = ((\lambda - v - 1/2)^2 - 1/4)\psi_{v,\lambda} = (-E_v - 1/4)\psi_{v,\lambda}. \tag{2.49}$$

Hence, for the Morse potential, the eigenvalues of the operator T_3 represent well depth parameters which can vary by integer amounts, while the eigenvalue of the Casimir of the algebra $so(2,1)$ determine the energy. Hence $so(2,1)$ representations correspond to Morse potentials at constant energy and this forms the basis of the so-called potential group approach to the Morse oscillator (Wu and Alhassid 1990, Englefield and Quesne 1991). Note that the present approach does not require the introduction of an additional (angular) variable. In addition, in contrast to the Coulomb case, in which the variable ρ is dependent upon the quantum number n (and hence the energy) through the scaling $\rho = Zr/n$, and acts as a space dilation operator, here the scaling involves the well depth parameter λ, which acts to displace the equilibrium position of the Morse potential via equation (2.42).

In the present case, when acting on the eigenstate $\psi_{v,\lambda}$, the transition operators $T_\pm$ have the form

$$T_\pm = \mp\left(\frac{d}{dx} - \frac{1}{2}\right) + \lambda - e^{-x} \tag{2.50}$$

where we have cancelled a term which vanishes identically under such circumstances. In addition, we have

$$T_3 T_\pm \psi_{v,\lambda} = T_\pm (T_3 \pm 1)\psi_{v,\lambda} = (\lambda \pm 1)T_\pm \psi_{v,\lambda}. \tag{2.51}$$

Since the eigenvalues of T_3 are displaced by unit steps, and since constant energy implies a constant value of $\lambda - v$, we must have

$$T_\pm \psi_{v,\lambda} \propto \psi_{v\pm1,\lambda\pm1}. \tag{2.52}$$

Furthermore

$$T_\mp T_\pm = T_3(T_3 \pm 1) - C_2 = \lambda(\lambda \pm 1) - (\lambda - v)(\lambda - v - 1). \tag{2.53}$$

Thus these operators act as *shift* operators for the Morse potential, acting to change the well depth parameter while conserving the energy. From equation (2.42), we note that the variable x also depends on the well depth parameter λ, so that we have generated a sequence of Morse oscillator potentials, displaced in both well depth and equilibrium separation. This will be discussed in more detail when considering supersymmetry and shape invariance for the Morse oscillator potential.

In summary, we have generated for the Morse potential shift operators which connect states of displaced Morse oscillators, with differing numbers of bound states and different equilibrium separations, at constant value of the energy. This is in contrast to the radial Coulomb and harmonic oscillator problems, where the operators act as ladder operators, changing the energy. This difference may be attributed to the fact that the Casimir of the algebra $so(2,1)$ for the Morse potential depends on the energy eigenvalue, and so is directly related to the Hamiltonian in this case. Hence we have generated the shift operators $S_\pm$ defined by

$$S_\pm = \mp \left(\frac{\mathrm{d}}{\mathrm{d}x} - \frac{1}{2} \right) + \lambda - \mathrm{e}^{-x} \tag{2.54}$$

such that

$$S_\pm \psi_{v,\lambda} = [\lambda(\lambda \pm 1) - (\lambda - v)(\lambda - v - 1)]^{1/2} \psi_{v\pm1,\lambda\pm1}. \tag{2.55}$$

3. Mappings between systems

It has long been known that the systems discussed above are connected, at least in a pairwise manner, and this is usually demonstrated by means of a similarity transformation on the relevant Schrödinger equation. In particular, the Morse oscillator has been related to the radial Coulomb problem (Morse 1929) and also to the two dimensional harmonic oscillator (Montemayor and Urrutia 1983) although it does not appear to have been explicitly related to the radial harmonic oscillator.

This section is designed to demonstrate the close relationship of these problems, within a common framework, and it will be demonstrated in a later section that these mappings can be extended to cover the radial Coulomb and radial harmonic oscillator problems in an arbitrary number of angular dimensions. We shall provide here an approach to mappings between each pair of systems which is based on the $so(2,1)$ realizations presented in section 2. Although we shall consider the mappings in a single direction only, it is obvious that the mappings are indeed reversible. We shall consider each pair in turn.

3.1. Coulomb–harmonic oscillator mapping

From the definitions of W_1 given in equations (2.4) and (2.27), we obtain the mapping

$$\rho \longrightarrow \tfrac{1}{2}\xi^2 \tag{3.1}$$

from which equation (2.7) becomes

$$W_2 = -\mathrm{i}\rho\frac{\mathrm{d}}{\mathrm{d}\rho} \longrightarrow -\frac{\mathrm{i}}{2}\xi\frac{\mathrm{d}}{\mathrm{d}\xi}. \tag{3.2}$$

Comparison with equation (2.30) shows that a similarity transformation of the form $\xi^\alpha W_2 \xi^{-\alpha}$ is required, implying $\alpha = -\tfrac{1}{2}$. Hence we apply the mapping (with accompanying similarity transformation)

$$W_2 \longrightarrow \tilde{W}_2 = \xi^{-1/2}W_2\xi^{1/2} = -\frac{\mathrm{i}}{2}\left(\xi\frac{\mathrm{d}}{\mathrm{d}\xi} + \frac{1}{2}\right). \tag{3.3}$$

Note that the transformed operators

$$\tilde{T}_i = \xi^{-1/2}T_i\xi^{1/2} \tag{3.4}$$

have precisely the same commutation relations as the T_i themselves.

A comparison of the eigenvalues of the operator T_3 in equations (2.10) and (2.31) permits the identification

$$n \equiv v + l + 1 \longrightarrow (2v + l + 3/2)/2 \tag{3.5}$$

so that we have

$$l \longrightarrow \tfrac{1}{2}l - \tfrac{1}{4}. \tag{3.6}$$

It is then a straightforward matter to confirm that the eigenvalues of the Casimir operator map consistently from $l(l + 1)$ to $\tfrac{1}{4}[l(l + 1) - \tfrac{3}{4}]$ as required. With the above change in l value, eigenstates of the radial Coulomb problem are mapped systematically on to those of the radial harmonic oscillator.

Finally, the transition operators $T_\pm$, given by equation (2.15), become, following the above identifications and the appropriate similarity transformation

$$T_\pm = \pm\rho\frac{\mathrm{d}}{\mathrm{d}\rho} - \rho + n \longrightarrow \frac{1}{2}\left[\pm\left(\xi\frac{\mathrm{d}}{\mathrm{d}\xi} + \frac{1}{2}\right) - \xi^2 + 2v + l + \frac{3}{2}\right] \tag{3.7}$$

as required by equation (2.33).

3.2. Radial harmonic oscillator–Morse oscillator mapping

In this case, from a comparison of equations (2.27) and (2.44) for W_1, we have

$$\tfrac{1}{2}\xi^2 \longrightarrow \mathrm{e}^{-x} \tag{3.8}$$

and application of this mapping to equation (2.30) for W_2 yields

$$W_2 = -\frac{\mathrm{i}}{2}\left(\xi\frac{\mathrm{d}}{\mathrm{d}\xi} + \frac{1}{2}\right) \longrightarrow \mathrm{i}\left(\frac{\mathrm{d}}{\mathrm{d}x} - \frac{1}{4}\right). \tag{3.9}$$

As in subsecion 3.1 above, by comparison of equations (2.47) and (3.9), we require a similarity transformation, which in this case is easily seen to give

$$W_2 \longrightarrow \bar{W}_2 = e^{x/4} W_2 e^{-x/4} = i\left(\frac{d}{dx} - \frac{1}{2}\right). \tag{3.10}$$

From a comparison of the eigenvalues of T_3, given in equations (2.31) and (2.48), we have

$$v + \tfrac{1}{2}l + \tfrac{3}{4} \longrightarrow \lambda$$

or, equivalently

$$\tfrac{1}{2}l \longrightarrow \lambda - v - \tfrac{3}{4} \tag{3.11}$$

where we note that l = constant is transformed into $\lambda - v$ = constant, which implies constant energy in the case of the Morse oscillator (see equation (2.40)). It is a straightforward matter to confirm the mapping of the Casimir operator C_2 (from equation (2.32))

$$C_2 = \left(\tfrac{1}{2}l - \tfrac{1}{4}\right)\left(\tfrac{1}{2}l + \tfrac{3}{4}\right) \longrightarrow (\lambda - v)(\lambda - v - 1) \tag{3.12}$$

as required by equation (2.49). Thus the eigenstates of the radial harmonic oscillator are systematically mapped on to degenerate eigenstates of a sequence of displaced Morse oscillators, with each value of l mapping on to a particular energy level of the sequence of Morse oscillators.

Finally, we can confirm from equations (2.33) and (2.50) that the transition operators $T_\pm$ (given by equation (2.33)) are transformed correctly as follows:

$$T_\pm = \frac{1}{2}\left[\pm\left(\xi\frac{d}{d\xi} + \frac{1}{2}\right) - \xi^2 + 2v + l + \frac{3}{2}\right] \longrightarrow \mp\left(\frac{d}{dx} - \frac{1}{2}\right) + \lambda - e^{-x} \tag{3.13}$$

using the appropriate similarity transformation, in agreement with equation (2.50).

3.3. Radial Coulomb–Morse oscillator mapping

In this case, from a comparison of equations (2.4) and (2.44) for W_1, we have

$$\rho \longrightarrow e^{-x} \tag{3.14}$$

from which equation (2.7) for W_2 transforms as

$$W_2 = -i\rho\frac{d}{d\rho} \longrightarrow i\frac{d}{dx}. \tag{3.15}$$

Comparison with equation (2.47) shows that we require here the similarity transformation

$$W_2 \longrightarrow \bar{W}_2 = e^{x/2} W_2 e^{-x/2} = i\left(\frac{d}{dx} - \frac{1}{2}\right). \tag{3.16}$$

From equations (2.10) and (2.48) for T_3, we have

$$n \equiv v + l + 1 \longrightarrow \lambda$$

or, equivalently

$$l \longrightarrow \lambda - v - 1 \tag{3.17}$$

from which we can verify the relation

$$C_2 = l(l+1) \longrightarrow (\lambda - v - 1)(\lambda - v) \tag{3.18}$$

in accordance with equations (2.12) and (2.49).

For the transition operators, $T_{\pm}$, we have the mapping (including similarity transformation)

$$T_{\pm} = \pm \rho \frac{\mathrm{d}}{\mathrm{d}\rho} - \rho + n \longrightarrow \mp \left(\frac{\mathrm{d}}{\mathrm{d}x} - \frac{1}{2} \right) + \lambda - \mathrm{e}^{-x} \tag{3.19}$$

as required from equations (2.15) and (2.50). Note again that constant l is mapped on to constant $\lambda - v$, which corresponds to constant energy in the case of the Morse oscillator. So we can systematically map the set of Coulomb eigenstates, for each value of the l quantum number, on to the set of degenerate eigenstates of a sequence of displaced Morse oscillators.

We have demonstrated that the various problems may be transformed into each other by a simple change of variable and accompanying similarity transformation, where the appropriate mappings may be easily identified from the common $so(2,1)$ algebraic formulation. In addition, the energy levels of each of the three problems can be sytematically mapped on to energy levels of the other two, provided that the Morse oscillator is generalized to encompass a set of displaced Morse oscillators, with well depth parameter shifted by unit steps, thereby changing the number of bound levels by one unit. This pattern is precisely that found in supersymmetric quantum mechanical treatments of the Morse oscillator (Sukumar 1985), where the supersymmetric partner potential to a given Morse oscillator has the same set of energy eigenvalues except that the ground level of the parent is absent. This will be discussed further in the next section.

4. Supersymmetric quantum mechanical treatment

In this section, we provide an alternative algebraic approach to the three problems, which serves to highlight further the close connection between them. We employ the method of supersymmetric quantum mechanics (Witten 1981, Sukumar 1985), which is equivalent (Alves and Drigho Filho 1988, Montemayor and Salem 1989) to the earlier factorization approach of Schrödinger (1940). Although some of the results discussed below are known, we shall present a summary here in order to connect with the $so(2,1)$ algebraic approach discussed in section 2.

4.1. Radial Coulomb problem

In contrast to our earlier discussion of $so(2,1)$ symmetry, we shall use the unscaled coordinate r, whereby the radial Schrödinger equation for the Coulomb problem has the form

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{l(l+1)}{r^2} - \frac{2Z}{r} - 2E_{v,l}\right]\psi_{v,l} = 0. \tag{4.1}$$

As before, v is the quantum number identifying the number of radial nodes.

We can factorize (4.1) in the form

$$a^{\dagger}(l)a(l)\psi_{v,l} = 2\Delta E_{v,l}\psi_{v,l} \tag{4.2}$$

where the ladder operators $a(l)$ and $a^{\dagger}(l)$ are given by

$$a(l) = \frac{\mathrm{d}}{\mathrm{d}r} - \frac{(l+1)}{r} + \frac{Z}{l+1} \tag{4.3}$$

and

$$a^{\dagger}(l) = -\frac{\mathrm{d}}{\mathrm{d}r} - \frac{(l+1)}{r} + \frac{Z}{l+1}. \tag{4.4}$$

The energy expression $\Delta E_{v,l}$ is given by

$$\Delta E_{v,l} = E_{v,l} - E_{0,l} = Z^2\left(-\frac{1}{2(v+l+1)^2} + \frac{1}{2(l+1)^2}\right) \tag{4.5}$$

and refers to the excitation energy from the ground level corresponding to a particular l value. We have assumed in the above that the energy eigenvalues are known, but it is a straightforward matter to determine these eigenvalues directly from the assumed factorization of the Hamiltonian.

The above operators can be shown to change the angular momentum quantum number at constant energy, thereby behaving as shift operators rather than ladder operators (Haymaker and Rau 1986, Ding 1987, Valance and Morgan 1990). The partner Hamiltonian $a(l)a^{\dagger}(l)$ can be expressed in the form

$$a(l)a^{\dagger}(l) = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{(l+1)(l+2)}{r^2} - \frac{2Z}{r} + \frac{Z^2}{(l+1)^2}$$
$$\equiv a^{\dagger}(l+1)a(l+1) + 2\Delta E_{1,l}. \tag{4.6}$$

The partner Hamiltonian corresponds to an increase in l quantum number of one unit, and has the same eigenvalue spectrum as the parent Hamiltonian except that the ground level of the parent potential is absent.

The action of the shift operators is as follows, where we assume normalized wavefunctions

$$a(l)\psi_{v,l} = \sqrt{2\Delta E_{v,l}}\,\psi_{v-1,l+1} \tag{4.7}$$

$$a^{\dagger}(l)\psi_{v-1,l+1} = \sqrt{2\Delta E_{v,l}}\,\psi_{v,l}. \tag{4.8}$$

Note that $a^\dagger(l)$ does *not* act on $\psi_{v,l}$ but on the result of operating on $\psi_{v,l}$ with $a(l)$, which is $\psi_{v-1,l+1}$. The operator which acts on $\psi_{v,l}$ to increase the number of radial nodes is $a^\dagger(l-1)$, given by

$$a^\dagger(l-1) = -\frac{d}{dr} - \frac{l}{r} + \frac{Z}{l} \tag{4.9}$$

such that

$$a^\dagger(l-1)\psi_{v,l} = \sqrt{2\Delta E_{v+1,l-1}}\,\psi_{v+1,l-1}. \tag{4.10}$$

In contrast to the $so(2,1)$ algebraic approach, the supersymmetric approach generates operators which transform degenerate states of the Coulomb problem into each other. We summarize the results for the Coulomb shift operators as follows:

$$S_-\psi_{v,l} \equiv \left(\frac{d}{dr} - \frac{(l+1)}{r} + \frac{Z}{l+1}\right)\psi_{v,l} = \sqrt{2\Delta E_{v,l}}\,\psi_{v-1,l+1} \tag{4.11}$$

$$S_+\psi_{v,l} \equiv \left(-\frac{d}{dr} - \frac{l}{r} + \frac{Z}{l}\right)\psi_{v,l} = \sqrt{2\Delta E_{v+1,l-1}}\,\psi_{v+1,l-1}. \tag{4.12}$$

4.2. Radial isotropic oscillator

In this case, the radial Schrödinger equation for the isotropic oscillator is (in reduced units)

$$\left[-\frac{d^2}{d\xi^2} + \frac{l(l+1)}{\xi^2} + \xi^2 - 2E_{v,l}\right]\psi_{v,l} = 0 \tag{4.13}$$

where v is the vibrational quantum number, l is the angular momentum quantum number, and the energy eigenvalues are given by

$$E_{v,l} = 2v + l + \tfrac{3}{2}. \tag{4.14}$$

Note that degeneracies involve eigenstates corresponding to either even or odd values of l in this case, the two sets being essentially disjoint.

We can factorize (4.13) in the form

$$a^\dagger(l)a(l)\psi_{v,l} = 2\Delta E_{v,l}\psi_{v,l} \tag{4.15}$$

where $\Delta E_{v,l} = E_{v,l} - E_{0,l}$ and where the shift operators have the form

$$a(l) = \frac{d}{d\xi} - \frac{(l+1)}{\xi} + \xi \tag{4.16}$$

and

$$a^\dagger(l) = -\frac{d}{d\xi} - \frac{(l+1)}{\xi} + \xi. \tag{4.17}$$

The partner potential $a(l)a^\dagger(l)$ then becomes

$$a(l)a^\dagger(l) = -\frac{d^2}{d\xi^2} + \frac{(l+1)(l+2)}{\xi^2} + \xi^2 - 2\left(l + \frac{1}{2}\right)$$
$$= a^\dagger(l+1)a(l+1) + 2(2) \equiv a^\dagger(l+1)a(l+1) + 2\Delta E_{1,l}. \qquad (4.18)$$

Thus the partner potential corresponds to an increase in l of one unit and has the same energy spectrum as the parent except for the absence of the lowest level. Although this situation appears to be qualitatively the same as in the Coulomb case we note that the energy spectrum of the partner, with l increased by one unit, has been shifted upwards by one unit to achieve the degeneracy required by supersymmetry. As we move from $l = 0$ to higher values of l via supersymmetry, the various energy level spectra are shifted upwards by l units for the partner labelled by angular momentum quantum number l. This is possible since the energy zero is not coupled to a dissociation limit in the present case.

From equations (4.15), (4.16) and (4.17), we can deduce the shift operator relations (assuming normalized eigenstates)

$$a(l)\psi_{v,l} = \sqrt{2\Delta E_{v,l}}\,\psi_{v-1,l+1} \qquad (4.19)$$

$$a^\dagger(l)\psi_{v-1,l+1} = \sqrt{2\Delta E_{v,l}}\,\psi_{v,l} \qquad (4.20)$$

or, equivalently

$$a^\dagger(l-1)\psi_{v,l} = \sqrt{2\Delta E_{v+1,l-1}}\,\psi_{v+1,l-1}. \qquad (4.21)$$

Note that these are actually quasi-shift operators, which act to change the l quantum number by ± 1, but *also* change the energy. The method of supersymmetric quantum mechanics offers no direct route to the determination of proper shift operators which should act at constant energy to change the l quantum number by ± 2. If we denote the above quasi-shift operators by $V_\pm$, their effect on normalized eigenstates is as follows:

$$V_-\psi_{v,l} \equiv \left(\frac{d}{d\xi} - \frac{(l+1)}{\xi} + \xi\right)\psi_{v,l} = \sqrt{2\Delta E_{v,l}}\,\psi_{v-1,l+1} \qquad (4.22)$$

$$V_+\psi_{v,l} \equiv \left(-\frac{d}{d\xi} - \frac{(l+1)}{\xi} + \xi\right)\psi_{v,l} = \sqrt{2\Delta E_{v+1,l-1}}\,\psi_{v-1,l+1}. \qquad (4.23)$$

4.3. Morse oscillator

It will be convenient here to use directly the Morse variable y as introduced in section 2. The Morse Schrödinger equation has the form

$$\left[-\frac{d^2}{dy^2} + \lambda^2 e^{-2y} - 2\lambda^2 e^{-y} - E_{v,\lambda}\right]\psi_{v,\lambda} = 0 \qquad (4.24)$$

where the energy eigenvalues are $E_{v,\lambda} = -(\lambda - v - 1/2)^2$. We factorize this equation in the form

$$a^\dagger(\lambda)a(\lambda)\psi_{v,\lambda} = \Delta E_{v,\lambda}\psi_{v,\lambda} \tag{4.25}$$

where $\Delta E_{v,\lambda} = E_{v,\lambda} - E_{0,\lambda}$. Then

$$a(\lambda) = \frac{d}{dy} + \lambda(1 - e^{-y}) - \frac{1}{2} \tag{4.26}$$

$$a^\dagger(\lambda) = -\frac{d}{dy} + \lambda(1 - e^{-y}) - \frac{1}{2}. \tag{4.27}$$

The partner Hamiltonian $a(\lambda)a^\dagger(\lambda)$ is expressible in the form

$$a(\lambda)a^\dagger(\lambda) = -\frac{d^2}{dy^2} + (\lambda - 1)^2\left[1 - \left(\frac{\lambda}{\lambda - 1}\right)e^{-y}\right]^2 + \lambda - \frac{3}{4} \tag{4.28}$$

or, equivalently, as

$$a(\lambda)a^\dagger(\lambda) = (a^\dagger(\lambda) - 1)(a(\lambda) - 1) + \Delta E_{1,\lambda}. \tag{4.29}$$

We note that the partner Hamiltonian corresponds to a displaced Morse oscillator corresponding to a unit shift in well depth parameter, and also to a shift in the equilibrium position of the oscillator. We note from equation (4.24) that the energy spectrum of the Morse oscillator is independent of the position of equilibrium. This is a particularly striking example of so-called shape invariance, as defined by Gendenshtein (1983), whereby the partner potentials are related to the parent by changes in certain parameters. A recent analysis of the concept of supersymmetry shape invariance in relation to solvability (Montemayor and Salem 1989) has demonstrated that its domain of application is precisely the same as that of the factorization method.

The operators $a(\lambda)$ and $a^\dagger(\lambda)$ are shift operators, acting at constant energy. For normalized eigenstates, we have

$$a(\lambda)\psi_{v,\lambda} = \sqrt{\Delta E_{v,\lambda}}\,\psi_{v-1,\lambda-1} \tag{4.30}$$

$$a^\dagger(\lambda)\psi_{v-1,\lambda-1} = \sqrt{\Delta E_{v,\lambda}}\,\psi_{v,\lambda} \tag{4.31}$$

or, equivalently

$$(a^\dagger(\lambda) + 1)\psi_{v,\lambda} = \sqrt{\Delta E_{v+1,\lambda+1}}\,\psi_{v+1,\lambda+1}. \tag{4.32}$$

The simple relationship between the shift operators for displaced Morse potentials has been exploited previously (Cooper 1992) in a treatment of annihilation operator coherent states of the Morse oscillator, whereby the ground-states of the various partner potentials are eigenstates of the operator $a(\lambda)$ with unit eigenvalues. This follows from the result $a(\lambda) \rightarrow a(\lambda) - 1$ as $\lambda \rightarrow \lambda - 1$.

If we now rewrite these equations in terms of the variable x introduced in the $so(2,1)$ algebraic approach in section 2, we have

$$a(\lambda) = \frac{d}{dx} - e^{-x} + \lambda - \frac{1}{2} \equiv S_- \tag{4.33}$$

$$a^\dagger(\lambda) + 1 = -\frac{d}{dx} - e^{-x} + \lambda + \frac{1}{2} \equiv S_+. \tag{4.34}$$

In addition, we have

$$\sqrt{\Delta E_{v,\lambda}} = [(\lambda - 1/2)^2 - (\lambda - v - 1/2)^2]^{1/2}$$
$$= [\lambda(\lambda - 1) - (\lambda - v)(\lambda - v - 1)]^{1/2} \tag{4.35}$$
$$\sqrt{\Delta E_{v+1,\lambda+1}} = [(\lambda + 1/2)^2 - (\lambda - v - 1/2)^2]^{1/2}$$
$$= [\lambda(\lambda + 1) - (\lambda - v)(\lambda - v - 1)]^{1/2}. \tag{4.36}$$

We observe that the shift operators S_+ and S_-, defined by equations (4.33) and (4.34) with their action given in equations (4.35) and (4.36), are indeed identical to those generated by the $so(2,1)$ algebraic approach, as given in equations (2.54) and (2.55).

5. Additional ladder and shift operators via mappings

In the case of the Coulomb problem, we have demonstrated that a combination of the $so(2,1)$ algebraic approach and that of supersymmetric quantum mechanics yield ladder and shift operators respectively. In the case of the radial isotropic oscillator, ladder operators are generated in the $so(2,1)$ algebraic approach whereas the operators generated by supersymmetry change both the l quantum number (by ± 1) and the energy. In the case of the Morse potential, the two approaches generate an equivalent set of shift operators. In this section, we will demonstrate that by use of the mappings identified in section 3, we can generate ladder operators for the Morse potential as well as true shift operators for the radial isotropic oscillator from the shift operators corresponding to the Coulomb potential.

5.1. Coulomb shift operators → oscillator shift operators

Here, as shown in section 3, we have the following mappings:

$$\rho \equiv Zr/n \longrightarrow \xi^2/2 \tag{5.1}$$

which gives the relations

$$r \longrightarrow \xi^2 \qquad \text{and} \qquad Z/n \longrightarrow \tfrac{1}{2}. \tag{5.2}$$

Also

$$n \equiv v + l + 1 \longrightarrow v + \tfrac{1}{2}l + \tfrac{3}{4} \tag{5.3}$$

so that

$$l \longrightarrow \tfrac{1}{2}l - \tfrac{1}{4}. \tag{5.4}$$

We note from equations (5.2) and (5.3) the further result

$$Z \longrightarrow (v + \tfrac{1}{2}l + \tfrac{3}{4})/2. \tag{5.5}$$

By use of the appropriate similarity transformation (equation (3.4)) the Coulomb shift operators (equations (4.11) and (4.12)) transform as

$$S_- = r^{-1/4}a(l)r^{1/4} = \frac{d}{dr} - \frac{l+3/4}{r^2} + \frac{Z}{l+1}$$
$$\longrightarrow \frac{1}{2}\left[\frac{1}{\xi}\frac{d}{d\xi} - \frac{l+1}{\xi^2} + 1 + \frac{4v}{2l+3}\right] \tag{5.6}$$

$$S_+ = r^{-1/4}a^\dagger(l-1)r^{1/4} = -\frac{d}{dr} - \frac{l+1/4}{r^2} + \frac{Z}{l}$$
$$\longrightarrow \frac{1}{2}\left[-\frac{1}{\xi}\frac{d}{d\xi} - \frac{l}{\xi^2} + 1 + \frac{4(v+1)}{2l-1}\right] \tag{5.7}$$

where we have performed the similarity transformation with respect to the original variable prior to converting to the new variable. The constants of proportionality are given by

$$\sqrt{(2\Delta E_{v,l})} \longrightarrow \frac{2}{2l+3}[v(v+l+3/2]^{1/2} \tag{5.8}$$

and

$$\sqrt{(2\Delta E_{v+1,l-1})} \longrightarrow \frac{2}{2l-1}[(v+1)(v+l+1/2)]^{1/2}. \tag{5.9}$$

The above results take no account of any changes due to normalization. We shall start from normalized oscillator wavefunctions and convert to Coulomb expectation values to see how normalized Coulomb wavefunctions should be modifed to yield normalized oscillator wavefuntions. Since $r = \xi^2$, then $d\xi = (r^{-1/2}/2)dr$. Also, from the similarity transformation $\bar{\psi} = r^{-1/4}\psi$ we have $\bar{\psi}^2 = r^{-1/2}\psi$ so that

$$\int \bar{\psi}^2 d\xi \longrightarrow \frac{1}{2}\int \frac{1}{r}\psi^2 dr \tag{5.10}$$

where the transformed (oscillator) wavefunctions are represented as $\bar{\psi}$. Note that radial eigenstates are involved in equation (5.10).

Since we are dealing with exact eigenstates, we can use the Hellmann–Feynman theorem (Hellmann 1937, Feynman 1939) in the form

$$\int r^{-1}\psi_{v,l}^2 dr = -\left\langle \frac{\partial H}{\partial Z} \right\rangle_{v,l} = -\frac{\partial E_{v,l}}{\partial Z} = \frac{Z}{n^2} \tag{5.11}$$

since $E_{v,l} = -Z^2/2n^2$. The shift operators act at constant n, so this result indicates that there is no further contribution from a change in normalization in this case.

Hence the relevant oscillator shift operators become

$$S_- \bar{\psi}_{v,l} = \frac{2}{2l+3}[v(v+l+3/2]^{1/2}\psi_{v-1,l+2} \tag{5.12}$$

$$S_+ \bar{\psi}_{v,l} = \frac{2}{2l-1}[(v+1)(v+l+1/2)]^{1/2}\psi_{v+1,l-2} \tag{5.13}$$

where $S_\pm$ are given by equations (5.6) and (5.7) respectively. The l quantum number of the oscillator changes by ± 2 when the l quantum number of the Coulomb system chages by ± 1, as a result of equation (5.4). Thus the Coulomb shift operators arising from supersymmetry may be mapped on to oscillator shift operators, which are not obtainable directly either by the use of $so(2,1)$ symmetry or by the method of supersymmetric quantum mechanics.

5.2. Coulomb shift operators → Morse ladder operators

Here, we shall use the Coulomb shift operators to generate ladder operators for a single Morse oscillator. The shift operators derived above connect degenerate states of displaced Morse oscillators. In the present case, we have (equation (3.14)) the mapping $\rho \equiv Zr/n \longrightarrow \lambda e^{-y}$ so that

$$r \longrightarrow e^{-y} \quad \text{and} \quad Z/n \longrightarrow \lambda \tag{5.14}$$

Since $n = v + l + 1 \longrightarrow \lambda$, then

$$l \longrightarrow \lambda - v - 1 \tag{5.15}$$

and, for consistency, using equation (5.14), we must also have

$$Z \longrightarrow \lambda^2. \tag{5.16}$$

Using the appropriate similarity transformation (equation (3.16)), the Coulomb shift operators (equations (4.11) and (4.12)) transform as

$$S_- = r^{-1/2}S_- r^{1/2} = \frac{d}{dr} - \frac{(l+1/2)}{r} + \frac{Z}{l+1}$$
$$\longrightarrow -e^y\frac{d}{dy} - (\lambda - v - 1/2)e^y + \frac{\lambda^2}{\lambda - v} \boxminus L_- \tag{5.17}$$

$$S_+ = r^{-1/2}S_+ r^{1/2} = \frac{d}{dr} - \frac{l+1/2}{r} + \frac{Z}{l}$$
$$\longrightarrow e^y\frac{d}{dy} - (\lambda - v - 1/2)e^y + \frac{\lambda^2}{\lambda - v - 1} \equiv L_+. \tag{5.18}$$

In this case, we also have

$$\sqrt{(2\Delta E_{v,l})} \longrightarrow \frac{\lambda}{\lambda - v}[v(2\lambda - v)]^{1/2} \tag{5.19}$$

$$\sqrt{(2\Delta E_{v+1,l-1})} \longrightarrow \frac{\lambda}{\lambda - v - 1}[(v+1)(2\lambda - v - 1)]^{1/2}. \tag{5.20}$$

In order to account for any change arising from normalization, we follow the same procedure as in subsection 5.1 above, to find

$$\int \bar{\psi}^2 \mathrm{d}\xi \rightarrow \int r^{-1}\psi^2 r^{-1} \mathrm{d}r = \int r^{-2}\psi^2 \mathrm{d}r. \tag{5.21}$$

If we write $L = l(l+1)$, we can again apply the Hellmann–Feynman theorem, giving

$$\int r^{-2}\psi_{v,l}^2 \mathrm{d}r = \left\langle \frac{\partial H}{\partial L} \right\rangle_{v,l} = \frac{\partial E_{v,l}}{\partial L} = \frac{Z^2}{(2l+1)n^3}. \tag{5.22}$$

Since the Coulomb shift operators act at constant n, the only term of relevance is the one depending on l, so that we need to incorporate $(2l+1)^{1/2}$ into $\psi_{v,l}$ to effect the necessary normalization, requiring a compensatory factor of $(2l+1)^{-1/2}$ in $\bar{\psi}_{v,l}$. Hence, the Morse ladder operators $L_{\pm}$ act on normalized Morse eigenstates as follows

$$L_{-}\bar{\psi}_{v,\lambda} = \left(\frac{2\lambda - 2v - 1}{2\lambda - 2v + 1}\right)^{1/2} \left(\frac{\lambda}{\lambda - v}\right)[v(2\lambda - v)]^{1/2}\bar{\psi}_{v-1,\lambda} \tag{5.22}$$

$$L_{+}\bar{\psi}_{v,\lambda} = \left(\frac{2\lambda - 2v - 1}{2\lambda - 2v - 3}\right)^{1/2} \left(\frac{\lambda}{\lambda - v - 1}\right)[(v+1)(2\lambda - v - 1)]^{1/2}\bar{\psi}_{v+1,\lambda}. \tag{5.23}$$

These ladder operators have been obtained previously from the factorization method (Huffaker and Dwivedi 1975), via transformation from the Coulomb problem, but the present approach is more direct and shows explicitly the origin of the various factors which appear in the final expressions. It is obvious that we could derive these ladder operators from the radial isotropic oscillator, starting from the true oscillator shift operators derived in subsection 5.1 above, by a similar treatment. It is worth noting that the relation between the two-dimensional harmonic oscillator and the Morse oscillator has also been used previously (Montemayor and Urrutia 1983, Berrondo *et al* 1987) to generate Morse ladder operators. We shall demonstrate in the next section that the one-dimensional Morse oscillator can be related to the radial Coulomb and radial harmonic oscillators in an arbitrary number of angular variables.

6. Generalization to D dimensions

We shall now consider a generalization of the above results to encompass radial Coulomb and radial harmonic oscillator potentials in an arbitrary number of angular dimensions. The topic of the D (or N) dimensional Schrödinger equation, in the context of large N expansions, has been reviewed recently (Chatterjee 1990) and we shall only consider here those results of direct relevance. For a particle moving in a spherically symmetric potential in D dimensions, the radial equation has the form (Louck 1960)

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{D-1}{r}\frac{\mathrm{d}}{\mathrm{d}r} + \frac{l(l+D-2)}{r^2} + V(r) - 2E_{v,l}\right]\psi_{v,l} = 0 \tag{6.1}$$

where $V(r)$ will correspond here to the Coulomb or harmonic oscillator potentials. If we perform a similarity transformation to reduce this equation to Schrödinger form by removing the term in d/dr, we find

$$\left[-\frac{d^2}{dr^2} + \frac{(D + 2l - 1)(D + 2l - 3)}{4r^2} + V(r) - 2E_{v,l} \right] \bar{\psi}_{v,l} = 0 \quad (6.2)$$

where $\bar{\psi}_{v,l} = r^{(D-1)/2}\psi_{v,l}$. The centrifugal term reduces to $l(l + 1)/r^2$ in the case $D = 3$, as required.

If we define the quantity

$$\Lambda = l + \frac{(D - 3)}{2} \quad (6.3)$$

then equation (6.2) can be written in the simple form

$$\left[-\frac{d^2}{dr^2} + \frac{\Lambda(\Lambda + 1)}{r^2} + V(r) - 2E_{v,\Lambda} \right] \bar{\psi}_{v,\Lambda} = 0 \quad (6.4)$$

such that the effective radial Schrödinger equation for dimension D has precisely the same form as for dimension $D = 3$, with the replacement of l by Λ, with Λ given by equation (6.3). The energy eigenvalues for the Coulomb and harmonic oscillator systems in D dimensions may be written down directly by comparison with the three-dimensional case.

We shall use this result to generalize the various mappings discussed in the earlier sections. The application of $so(2,1)$ algebra (Čížek and Paldus 1977) and the application of the methods of supersymmetric quantum mechanics (Alves and Drigo Filho 1988) to the Coulomb and oscillator problems have both included this generalization, but not in the context of relations between the systems themselves. We shall consider here in addition the relation between these generalized problems and the one dimensional Morse oscillator potential, by means of generalizations to the mappings discussed previously. Since the resultant generalizations are provided by the replacement of l by Λ, as given by equation (6.3), only a summary of the results will be provided here, since the derivations are equivalent to those discussed in earlier sections.

6.1. Radial Coulomb problem

Following equations (2.23) and (2.24), the generalized Coulomb ladder operators act on normalized eigenstates as

$$L_{\pm}\psi_{v,\Lambda} \equiv \left(\pm\rho\frac{d}{d\rho} - \rho + n \right)\psi_{v,\Lambda} = \left(\frac{n \pm 1}{n} \right)[n(n \pm 1) - \Lambda(\Lambda + 1)]^{1/2}\psi_{v\pm 1,\Lambda} \quad (6.5)$$

where $n = v + \Lambda + 1 \equiv v + l + 1 + (D - 3)/2$. Following equations (4.11) and (4.12), the generalized Coulomb shift operators act on normalized eigentates as

$$S_{-}\psi_{v,\Lambda} \equiv \left(\frac{d}{dr} - \frac{(\Lambda + 1)}{r} + \frac{Z}{\Lambda + 1} \right)\psi_{v,\Lambda} = \sqrt{2\Delta E_{v,\Lambda}}\,\psi_{v-1,\Lambda+1} \quad (6.6)$$

$$S_{+}\psi_{v,\Lambda} \equiv \left(-\frac{d}{dr} - \frac{\Lambda}{r} + \frac{Z}{\Lambda} \right)\psi_{v,\Lambda} = \sqrt{2\Delta E_{v+1,\Lambda-1}}\,\psi_{v+1,\Lambda-1} \quad (6.7)$$

where $E_{v,\Lambda} = -Z^2/2(v + \Lambda + 1)^2$ and $\Delta E_{v,\Lambda} = E_{v,\Lambda} - E_{0,\Lambda}$.

6.2. Radial harmonic oscillator problem

Following equations (2.36) and (2.37), the generalized oscillator ladder operators act on normalized eigenstates as

$$L_\pm \psi_{v,\Lambda} \equiv \frac{1}{2}\left(\pm\xi\frac{d}{d\xi} \pm \frac{1}{2} - \xi^2 + 2v + \Lambda + \frac{3}{2}\right) = [T_3(T_3+1) - C_2]^{1/2}\psi_{v\pm1,\Lambda} \quad (6.8)$$

where

$$T_3 = \tfrac{1}{2}(2v + \Lambda + 3/2)$$

and

$$C_2 = \tfrac{1}{4}[\Lambda(\Lambda+1) - 3/4]$$

with $\Lambda = l + (D-3)/2$.

The quasi-shift operators, given by equations (4.22) and (4.23), which act to change the representation and also the energy, are generalized as follows:

$$V_- \psi_{v,\Lambda} = \left(\frac{d}{d\xi} - \frac{(\Lambda+1)}{\xi} + \xi\right)\psi_{v,\Lambda} = \sqrt{2\Delta E_{v,\Lambda}}\,\psi_{v-1,\Lambda+1} \quad (6.9)$$

$$V_+ \psi_{v,\Lambda} = \left(-\frac{d}{d\xi} - \frac{\Lambda}{\xi} + \xi\right)\psi_{v,\Lambda} = \sqrt{2\Delta E_{v+1,\Lambda-1}}\,\psi_{v+1,\Lambda-1} \quad (6.10)$$

where $E_{v,\Lambda} = 2v + \Lambda + 3/2$ in this case.

From section 5.1, the Coulomb shift operators themselves are generalized as follows:

$$S_- \psi_{v,\Lambda} \equiv \frac{1}{2}\left[\frac{1}{\xi}\frac{d}{d\xi} - \frac{(\Lambda+1)}{\xi^2} + 1 + \frac{4v}{2\Lambda+3}\right]\psi_{v,\Lambda}$$

$$= \frac{2}{2\Lambda+3}[v(v+\Lambda+3/2]^{1/2}\psi_{v-1,\Lambda+2} \quad (6.11)$$

$$S_+ \psi_{v,\Lambda} \equiv \frac{1}{2}\left[-\frac{1}{\xi}\frac{d}{d\xi} - \frac{\Lambda}{\xi^2} + 1 + \frac{4(v+1)}{2\Lambda-1}\right]\psi_{v,\Lambda}$$

$$= \frac{2}{2\Lambda-1}[(v+1)(v+\Lambda+1/2]^{1/2}\psi_{v+1,\Lambda-2}. \quad (6.12)$$

We note in addition that the above results are valid for the Coulomb and harmonic oscillator systems in any number of angular dimensions, and the various mappings discused in section 3 are also applicable to these generalized systems, with the replacement of l by Λ. Specifically, the one dimensional Morse oscillator may be mapped on to the radial Coulomb and oscillator problems in any number of angular dimensions, and vice versa.

7. Conclusions

We have demonstrated that three of the well known exactly solvable potentials coorespond to diferent realizations of the algebra $so(2,1)$, and that a comparison of

the generators of the algebra may be used to identify mappings between each pair of systems. The $so(2, 1)$ transition operators act as ladder, or energy changing operators, in the case of the Coulomb and harmonic oscillator systems, and as shift operators, acting at constant energy, in the case of the Morse potential, in consequence of the fact that the latter Hamiltonian is expressible solely in terms of the Casimir operator of the algebra. Use of the methods of supersymmetric quantum mechanics, or factorization, permit the construction of shift operators for the Coulomb problem, and quasi-shift operators for the harmonic oscillator, which change the energy as well as the angular momentum quantum number. The shift operators for the Morse oscillataor as determined by supersymmetric quantum mechanics are equivalent to those arising from the $so(2, 1)$ algebraic treatment. By use of the mappings between the various systems, the Coulomb shift operators can be used to generate ladder operators for the Morse potential and true shift operators for the harmonic oscillator potential. These results can be extended to encompass radial Coulomb and oscillator problems in an arbitrary number of angular dimensions. In particular, we have demonstrated that the one dimensional Morse oscillator can be mapped both to and from radial Coulomb and oscillator problems in any number of angular dimensions, and these two radial problems can be mapped to and from each other.

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